Heterogeneous Intermediaries and Stability in Financial Networks[☆,☆☆]

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Abstract

I study a financial network in which banks are interconnected through cross-deposits they make in other banks. The liquidity demands these banks face are negatively correlated. Banks choose cross-deposits to insure themselves against actual realizations of their liquidity demands so to offer the best consumption schedule for their consumers. However, they may also be hit by an uninsurable excess liquidity demand. Banks liquidate their investment portfolios so to serve the excess demand they are faced with. I show that in such a framework banks that are already facing a high liquidity demand are more likely to incur the burden of excess liquidity shocks even when that shock has not directly hit them, i.e. relatively safer banks strategically pass liquidation costs to relatively less safe banks. I also show that private bailouts arise endogenously in this framework. If the strategic behavior of a bank results in the other bank's failure, the first bank may choose to incur the burden of the liquidity shock by itself to let the other bank survive and, thus, to control the indirect costs of failure feeding back to its portfolio. Therefore, there is no complete pecking order in banks' liquidation decisions. I also show that for some economies the financial network becomes more stable as the level of cross-deposits is increased from the minimum level that fully insures banks against liquidity demand uncertainty up to a threshold level.

Keywords: heterogeneity, financial networks, fragility, contagion, private bailouts, strategic withdrawals

1. Introduction

I study a financial network in which consumers are ex-ante similar. But they realize their types later and become either patient or impatient. Patient consumers are indifferent between consuming early or late. However, impatient

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consumers do not value late consumption and only consume in the early period. The role of financial intermediaries is to accept deposits from these consumers and provide consumption smoothing for them. the consumer-bank relationship in my model is similar to that of Diamond and Dybvig (1983).

Despite Diamond and Dybvig (1983), I allow for uncertainty regarding the fraction of impatient consumers in economy. I consider an economy with different regions. Each region realizes either a high or low fraction of impatient consumers among all its depositors after all intermediaries have made their investment decisions. This notion is similar to that in Allen and Gale (2000). They study a financial network in which not only consumers are insured against their liquidity demand uncertainty by banks, but banks are also insured against their liquidity needs by other banks. They consider a case in which an excess liquidity demand shock hits one bank in an state occurring with probability zero when no liquidity demand uncertainty exists. In other words, banks are either hit by an excess liquidity demand shock or liquidity demand uncertainty. They show this network is vulnerable to excess liquidity shocks and if this shock results in failure of the bank to which the shock hits, then it can spread throughout the network and affect other banks. Despite Allen and Gale (2000), I allow excess liquidity demand shock and liquidity demand uncertainty to independently hit a region. Not allowing for this to happen is equivalent to arguing that banks withdraw their cross-deposits such that they fully cancel out each others' deposits independent of the excess liquidity demand hitting them. I formally model this argument by considering an excess liquidity shock hitting different banks with different current liquidity needs. This setup lets me study heterogeneous responses of banks to excess liquidity shocks as a function of their liquidity demand uncertainty realizations. Eisenberg and Noe (2001) show that there exists a market clearing payment vector that solves all intermediaries optimization problems. This payment vector is such that some intermediaries default and the others survive. I find the equilibrium payment vectors in all states of my model.

I show that banks that are hit by excess liquidity shocks choose different liquidation strategies to procure the liquidity they need. In fact, they strategically make their withdrawal decisions so to maximize the welfare of consumers in their regions- potentially- at the cost of welfare of consumers in the other region. if the bank is unsafe ex-post, i.e. has drawn a high liquidity demand, it will liquidate its long-term investments to meet the liquidity demand shock and ,thus, incurs the additional costs by itself. However, if it is safe ex-post, i.e. has drawn a low realization of liquidity demand, it does not necessarily incur the costs of early liquidation. In fact, it can pass the costs to the unsafe bank by withdrawing what it has deposited in that.

Acharya et al. (2012) consider a model in which liquidity transfers between banks occur in two markets, the interbank lending market and the asset sales market. Banks investments are risky. Their investments may also require some refinancing in the middle period and, thus, they may face liquidity needs. They argue banks with surplus liquidity strategically underprovide lending to banks with liquidity needs so to induce them to inefficiently sell their assets. I show that even when banks cannot benefit from buying the at-risk banks' assets, they may still decide to withdraw their deposits in the unsafe bank and make that bank default, a notion similar to lending underprovision in Acharya et al. (2012).

I also show that safe banks are concerned about unsafe banks' stability as the failure of unsafe banks can spread through the network and negatively affect other banks' asset values. Thus, if passing all the costs to the unsafe bank causes its failure, i.e. if the shock is large enough, the safe bank may choose not to do so to let the unsafe bank survive. This is equivalent to a private bailout from the safe bank to the unsafe one. This result is consistent with that of Leitner (2005) in which he also shows occurrence of private bailouts.

Acemoglu et al. (2015) show that the interconnectedness can help a system absorb small shocks. however, it serves as a propagation mechanism for large shocks leading to a more fragile financial system. I show that the strategic decisions of financial intermediaries can also contribute to the fragility of a financial system beyond the size of shocks.

In this paper the cross-deposits that banks have made in the initial period, provides an extra liquidity source for ex-post safe banks. Thus, when they are hit by excess liquidity demand shock, they have the option of withdrawing their cross-deposits in addition to liquidating their long-term investments. Consistent with Allen and Gale (2000), I assume excess liquidity demand shocks hit the economy with probability zero. Therefore, banks do not consider this ex-post difference when making cross-deposits and the hold-up problem does not arise in their investment decisions. Nonetheless, there are some investment portfolios that yield the first-best consumption schedule in the states happening with nonzero probabilities, but the financial network in which they are implemented are different in terms of stability. Specifically, I show that a too small cross-deposits level cannot provide full insurance against liquidity demand uncertainty, and a too large level makes the financial system more fragile. Thus, a moderate level is optimal. I also show that a higher level of cross-deposits has a disciplinary effect on ex-post safe banks. It increases the spillover costs if the other bank defaults. Thus, the safe bank engages in more private bailouts and is more willing to help the other bank survive. Therefore, the optimal level of cross-deposits is larger than what is necessary to fully insure banks against liquidity demand uncertainty.

The differential impacts of network structure on systemic risk has motivated researchers to study the optimal and equilibrium network structures and shed light on the fundamental reasons that financial networks are formed the way they are. Stiglitz (2010) states a similar problem and considers the systemic risk arising from over-connected financial networks. He, thus, proposes a financial architecture with different regions in which each region constitutes competitive intermediaries. However, connections between regions are kept at a minimum to prevent the contagion effects in an event of crisis. In my model, the amount of cross-deposits banks make in each other improves total welfare as it is increased up to a threshold. However, it has no positive effect on total welfare as it passes that threshold and it results in even higher fragility as it is increased further. Therefore, my results are inline with Stiglitz (2010) arguments and in addition to his argument about existence of financial connections not necessarily stabilizing the financial system, show the magnitude of those connections have similar impacts. Babus (2016) considers a financial system similar to that of Allen and Gale (2000) and studies the network formation motives of intermediaries. She considers the intra-region connections of intermediaries as given and studies the equilibrium network structures within a region. She shows that endogenous network structures arise in which intermediaries share their losses through the connections they have formed. Farboodi (2017) also studies the emergence of financial networks and shows that the familiar star network is indeed an equilibrium network even though it is not efficient. Egan et al. (2017) consider a model with heterogeneous banks. However, they do not study the network implications, but competition between banks. Chang (2018) studies a collateralized debt network in which the lender defaults. That work can be in part connected to mine as in my work the bank who withdraws the deposits makes the other bank default. However, the question he studies and the model he uses are completely different with those of mine.

2. Model

The model has three time periods. There exists an ex-ante identical continuum of consumers in each region endowed with one unit of wealth in period zero. Consumers are either patient or impatient. Consumers do not know their types in period zero when they deposit their endowments in banks. The utility of consumers conditional on their types is the following,

$$U(c_1, c_2) = \begin{cases} u(c_1) & \text{if impatient} \\ u(c_1 + c_2) & \text{if patient} \end{cases}$$

in which function $u(.)$ is assumed to be twice continuously differentiable, increasing and strictly concave. c_1 is consumers' period one consumption and c_2 is their period two consumption. Thus, impatient consumers always need their endowments in period one. However, patient consumers can wait until period two if they are better off. If not, they join impatient consumers and consume in period one.

The economy has one long-term investment technology that returns $R > 1$ units in period two for each unit of endowments invested in period zero. However, premature liquidation of this investment is costly and will return $r \leq 1$ units if liquidated in period one. This investment technology is perfectly divisible. There is also a storage technology that transfers funds across periods without any costs, and any gains. A financial system exists in this economy that offers liquidity insurance against being impatient through a deposit contract such that consumers who become impatient in period one can consume more than one unit in period one by sacrificing what they would have consumed in period two if they became patient. consumers' types are private information, however total fraction of impatients is common knowledge. All consumers simultaneously make their deposits in period zero. all consumers choosing to withdraw early (either patient or impatient) do that simultaneously and all consumers withdrawing late do that simultaneously. the only available deposit contract is demand deposit, so consumers can withdraw at anytime they want. All banks respect the terms of contracts unless they are unable to and default.

There are two financial regions A and B in the economy. The banking system in each region is fully competitive. Hereafter, I assume that a representative bank exists in each region.[1] Total fraction of impatients in the entire economy is a known parameter π. However, similar to Allen and Gale (2000), the actual fraction of impatient consumers in each region is not ex-ante known. In the two equally likely states each happening with probability $p = 0.5$, The fraction of impatients in region A is$\pi - \omega$ in state S_1 and $\pi + \omega$ in state S_2. Since there is no aggregate uncertainty about total fraction of impatients in these states, the fraction of impatients in region B immediately follows. Hereafter, the risk arising from this uncertainty of liquidity demand is called diversifiable risk. Banks can make a deposit in each other the same way, and under similar terms, that consumers make deposits in banks. By doing so, banks can insure themselves against liquidity shocks arising from diversifiable risk.

There are also states occurring with zero probability in which an excess liquidity demand shock hits a region, and total fraction of impatients is more than π. This shock may hit a region that is currently facing a high liquidity demand $(\pi + \omega)$ or low liquidity demand $(\pi - \omega)$. It can also have different magnitudes. Hereafter, I call these states the unconventional states and the risk arising from the excess liquidity demand shock in these states the nondiversifiable risk. As these states occur with zero probability, they do not affect the pre-deposit game of banks and consumers.

My model has an important difference with that of Allen and Gale (2000). Allen and Gale (2000) assume that unconventional states occur in absence of diversifiable liquidity demand shock ω or equivalently, the cross-deposits between banks simply cancel out in unconventional states even before the nondiversifiable risk hits the economy. It can also be seen as a framework in which banks first observe their diversifiable liquidity demand shocks and decide what to do with their cross-deposits based on that, and only after that they observe should a nondiversifiable liquidity shock hits them. Thus, a bank's decision on how to handle diversifiable liquidity demand shock is not affected by the nondiversifiable liquidity demand shock hitting it. I relax this strong assumption, allow a shock to hit banks with different actual liquidity needs, and investigate banks' heterogeneous responses to that shock. A full characterization of states is shown in table 1.

2.1. Timing of the Model

Consumers are initially- in period zero- endowed with one unit of cash. Banks in each region offer a contract- to be specified- to consumers in that

[1]For simplicity, I'll refer to the representative bank in region A (B) with male (female) pronouns. I'll also use things' pronouns whenever region does not matter.

State	Probability	Region A	Region B
S_1	0.5	$\pi - \omega$	$\pi + \omega$
S_2	0.5	$\pi + \omega$	$\pi - \omega$
S_3	0	$\pi - \omega$	$\pi + \omega + \epsilon$
S_4	0	$\pi - \omega + \epsilon$	$\pi + \omega$
S_5	0	$\pi + \omega + \epsilon$	$\pi - \omega$
S_6	0	$\pi + \omega$	$\pi - \omega + \epsilon$

Table 1: Different states of the game, their probability of occurrence, and the fraction of impatient consumers in different regions in each state

region. Consumers choose whether to accept the terms of this contract and deposit in a bank in their region or go to autarky. Once depositing decisions are made, a contract pinning down the entire investment portfolio of banks- including the amount invested in long-term investment, the amount stored in storage technology, and the cross-deposits between banks- is proposed to banks in both regions. Banks in each region choose whether to accept this contract or not. If at least one of the banks does not accept the contract terms, none of the banks can make cross-deposits in other banks.

In period one, the state of economy is observed. Consumers privately observe their types as well. They, then, simultaneously choose whether to withdraw early or not. Each bank observes the decision of consumers and simultaneously chooses how to update its investment portfolio- including how much to withdraw from cross-deposits it has made- to meet the liquidity demand given the state of economy and considering the other bank's strategy. If each bank is able to incentive compatibly[2] keep its promised level of payment in period one specified in the contract, it serves the consumers withdrawing early with that specified amount. If not, the bank defaults, liquidates all its assets, and equally distributes them among all depositors in period one. If it survives, it harvests its remaining long-term investments and withdraws its remaining cross-deposits. It, then, equally distributes all the remaining resources among remaining patient consumers in the region and the remaining deposits of other banks.

The contract between bank and consumers in each region is of demand-deposit type. That is, consumers can withdraw their deposits whenever they want. The contract specifies the promised level of consumption in period one, C_1, and the promised level of consumption in period two C_2.[3] If a consumer wants to withdraw early, she is promised to receive C_1 and if she withdraws late, she is promised to receive C_2. Repayment of these promised values is, however, conditional on the observed state. If, there are more consumers withdrawing

[2]Bank should be able to serve early withdrawals with the promised level of consumption in period one and serve late withdrawal with a consumption level at least as good as that of period one so to avoid a run occurrence.

[3]Later, I will derive the actual consumption levels that are not necessarily similar to the promised ones.

early than what is expected, banks serve them all with C_1 only if they are still able to offer a higher amount than C_1 in period two to consumers withdrawing late so that the incentive compatibility condition of patient consumers is satisfied. Otherwise, patient consumers are better of joining impatient consumers and withdrawing in period one and banks will have to liquidate all their projects prematurely and distribute them among all depositors in period one. Thus, if a bank is not able to keep its promise, then it liquidates all its assets in period one and equally distributes them among all depositors.

The contract between banks specifies the amount of liquidity to be stored in storage technology, the amount of long-term investment and the amount of cross-deposits between banks. Note that the cross-deposits of each bank in the other bank should be equal so to satisfy the resource constraints of both banks.

2.2. The Optimal Consumption Schedule

Consumers are all ex-ante similar. We first consider the first-best consumption schedule that a social planner can offer to these consumers in period zero. The first-best allocation maximizes the expected ex-ante utility of a representative consumer before she observes her type. Since nondiversifiable liquidity demand shock ϵ occurs with zero probability, it does not affect the optimal portfolio designed ex-ante prior to realization of these shocks. On the other hand, the diversifiable liquidity demand shock ω can potentially change the characteristics of the optimal contract. Nonetheless, the availability of cross-depositing option makes it possible to fully insure banks and consumers against this risk and ,thus, let the financial system achieve the first-best consumption schedule. As there is no aggregate uncertainty about total fraction of impatients- in states happening with nonzero probability, the first best allocation is the same as that of Diamond and Dybvig (1983).[4] Thus, the optimal allocation is in a setting in which neither diversifiable nor nondiversifiable risk are present and solves the following problem,

$$\begin{aligned} \max_{C_1, C_2} \quad & \pi u(c_1) + (1-\pi)U(c_2) \\ s.t. \quad & \pi c_1 + \frac{(1-\pi)c_2}{R} = 1 \end{aligned}$$

Suppose the consumption schedule $(c_1, c_2) = (C_1, C_2)$ solves the social planner's optimization problem. Diamond and Dybvig (1983) show that the aforementioned consumption schedule is incentive compatible $C_1 \leq C_2$.[5] Thus, a patient consumer does not benefit if she decides to withdraw early while

[4]Strictly speaking, as I have not yet shown the first-best allocation is feasible in my model, the first-best allocation in our framework is weakly worse than that of Diamond and Dybvig (1983) in terms of expected ex-ante utility level of consumers.

[5]Note that incentive compatibility holds assuming the fraction of impatient consumers is known and that consumers play their types. If the fraction of impatient consumers deviates from π or if consumers do not play their types, incentive compatibility is not guaranteed.

other patient consumers wait until period two. In the next section I will show that investment portfolios exist that result in the aforementioned consumption schedule. Thus, the optimal contract between banks and consumers promises (C_1, C_2).

2.3. Characterization of the Optimal Contract Between Banks

Banks collect deposits from consumers and other banks. They can invest those resources in long-term investment technology whose return is high if harvested in period two but low if harvested prematurely, in storage technology which transfer resources across periods, or deposit them in other banks. Define x to be the amount stored in storage technology, y to be the amount of long-term investment, and z to be the amount deposited in other banks. Since all banks are ex-ante similar, they all cross-deposit the same amount of z and invest similar values of x and y. Thus, total resources available in a bank is $1 + z$ in which 1 is total consumers' deposits and z is deposits of other banks in this bank. The resource constraint of banks is the following,

$$\begin{aligned} x + y + z &\leq 1 + z \\ x + y &\leq 1 \end{aligned}$$

I will show that only investment portfolios of the following characteristics achieve the optimal consumption schedule. An optimal portfolio of a bank is then $(x, y, z) = (X, Y, Z)$,

$$\begin{aligned} X &= \pi C_1 \\ Y &= (1 - \pi)\frac{C_2}{R} \\ Z &\geq \omega \end{aligned} \tag{1}$$

First, I argue all the aforementioned conditions in (1) are necessary to yield the optimal consumption schedule. If $x < \pi C_1$, since both banks are ex-ante identical, total available liquidity in the economy in period one will be $2x < 2\pi C_1$. Whereas, Total fraction of impatient consumers in both regions is 2π and total liquidity needed to serve withdrawals in period one is $2\pi C_1$. Thus, total available liquidity is not enough and at least one bank has to prematurely liquidate its long-term investments, a process that is costly and, thus, prevents reaching the economy to the optimal consumption schedule. Thus, we should have $x \geq \pi C_1$. A similar argument can be addressed for the impossibility of meeting total second period withdrawal demands in both regions if $y < (1 - \pi)\frac{C_2}{R}$. Thus, we should have $y \geq (1 - \pi)\frac{C_2}{R}$.

Summing up the latter result with the former returns $x + y \geq \pi C_1 + (1 - \pi)\frac{C_2}{R} = 1$. However, resource constraint imposes the condition $x + y \leq 1$. The only solution that satisfies both inequalities is $x = X$ and $y = Y$.

Now assume $z < \omega$. Without loss of generality, consider a bank in region A and let the state be S_2. This bank cannot meet the entire liquidity demand in period one, i.e. $(\pi + \omega)C_1$, even if he uses all his available liquidity, i.e. πC_1, and withdraws all his deposits in banks in region B, i.e. $zC_1 < \omega C_1$. Therefore, he has to undertake the costly liquidation process of his long-term investments and is no longer able to offer the optimal consumption schedule. As a result, it is necessary for the optimal contract to have $z \geq \omega$.

Second, I argue that all the contracts that satisfy conditions (1), yield the optimal consumption schedule. In other words, conditions (1) are sufficient to reach the optimal consumption schedule. Suppose consumers have made their deposits and banks have chosen the investment portfolios consistent with conditions (1) in period zero. In period one, consumers privately observe their types and all players observe state of the economy. The optimal contracts and the optimal investment portfolios are already chosen. So banks can only decide how to manage the withdrawals while maintaining their portfolios. Consumers who happened to be patient, can choose whether to withdraw their deposits in period one or period two.[6]

Assume the realized state is S_1. Bank A is faced with a low liquidity demand $\pi - \omega$, while bank B is faced with high liquidity demand $\pi + \omega$. Bank A and B each have enough resources to meet the liquidity demand of a fraction π of consumers. Thus, bank A has extra liquid resources, while bank B needs extra liquid resources.

The situation is the opposite in the second period. Bank A is faced with a fraction of $(1 - \pi + \omega)$ of patient investors withdrawing in period two, while he has enough long-term investments available to distribute in period two for only the fraction of $(1 - \pi)$. On the other hand, bank B is faced with a fraction of $(1 - \pi - \omega)$ of patient investors withdrawing in period two, while he has enough long-term investments available to distribute in period two for fraction of $(1 - \pi)$. Thus, bank A needs extra resources in period two, while bank B has extra resources in period two.

Taking those reciprocal resource needs into consideration, bank B withdraws a fraction of her deposits in bank A in period one receiving the total of ωC_1 and bank A withdraws a fraction of his deposits in bank B in period two receiving the total of ωC_2. Bank A has $X = \pi C_1$ units of liquid assets in period one and now it receives another ωC_1 units. Thus, bank A can meet the demand of the fraction $(1 - \pi + \omega)$ in period one. In period two, bank A harvests his $Y = (1-\pi)C_2$ units of long-term investments, pays bank B total amount of ωC_2 in period two and pays the total of $(1 - \pi - \omega)C_2$ to the remaining consumers withdrawing in period two. Thus, all the consumers in region A consume the first-best consumption schedule of (C_1, C_2). Bank B can also offer the first-best consumption schedule by following the same argument and withdrawing her deposits in period two. The remaining deposits of banks in each other, i.e.

[6]Impatient consumers do not have a choice. They only receive utility if they withdraw in period one.

$Z - \omega$, are withdrawn by both sides in period one and thus, fully cancel out each other and have no net effects.

Now consider state S_2. The argument in this state is also similar to that of state S_1 with the difference being that bank B withdraws her deposits in period one and bank A withdraws his deposits in period two. Following the same argument, both banks achieve the first best consumption schedule.

Thus, consumers in both banks can achieve the optimal consumption schedule in states S_1 and S_2, even though their actual liquidity needs were initially unknown. Since states S_1 and S_2 are the only states occurring with nonzero probabilities, they are the only relevant states to design the ex-ante optimal contract. Therefore, the investment portfolios consistent with conditions (1) are optimal for both banks and yield the first best consumption schedule for all consumers.

Note that if $Z > \omega$, the additional cross-deposits above ω do not improve economy's total welfare. Any attempt by one side to withdraw $Z - \omega$ of deposits above what they are expected to withdraw, i.e. ω in state S_1 and 0 in state S_2 for bank B, is exactly retaliated by the other side and these two opposite withdrawals cancel each other out. However, the level of deposits is not neutral when one side defaults. Specifically, the consumers in failing bank's region benefits from higher cross-deposit levels, while the consumers in the other region incur a loss. I will discuss this issue in a more detailed manner in subsection (4.1). All the discussions up until that section assume $Z = \omega$.

3. Banks' Withdrawal Decisions in Unconventional States

In this section, I discuss the banks' strategies and their equilibrium responses in unconventional states. Analyses of states S_5 and S_6 are respectively similar to those of states S_3 and S_4. Thus, I will only analyze the latter two

The departing difference between these unconventional states and states S_1 and S_2 is that in the latter, aggregate liquidity demand is exactly equal to aggregate liquid investments. While in the former, aggregate liquidity demand exceeds aggregate liquid investments. Thus, partial liquidation of long-term investments is inevitable to meet total liquidity demand. Nonetheless, banks can choose how to manage their portfolios and which investments to liquidate. In other words, conditional on the state and the contracts, they choose whether to use their available liquid assets, liquidate their long-term investments, withdraw their deposits in the other bank, or a combination of them. The withdrawal of deposits in a bank has consequences on its response. Therefore, a shock hitting one bank can spread to another bank connected to it. The change in other bank's liquidation decision will also change the value of its deposits and, thus, feeds back to its depositors' portfolios including the first bank. Banks consider these spillover costs when making a decision regarding their portfolio management.

In all these unconventional states, one bank is faced with nondiversifiable liquidity demand shock in excess to what the financial system can insure. The affected bank liquidates some of its investments to provide the excess liquidity needed. It is also a possible choice for a bank to strategically collect more liquidity than needed by withdrawing its deposits in the other bank or liquidating its long-term investments. However, it is never optimal to do so. If it liquidates its long-term investments while it does not need the excess liquidity, it should transfer those prematurely liquidated investments to period two. An action that is costly and inefficient compared to waiting until period two to harvest long-term investments. Furthermore, if it withdraws its deposits in the other bank above the liquidity level it needs, it should transfer those resources to period two. Thus, the value of her liquidated deposits in period two equals period one withdrawal value of a deposit which due to incentive compatibility of the contracts is less than the value of period two withdrawals. Moreover, period one withdrawals make the other bank incur some costs which will eventually affect the first bank as well. Thus, the strategic over-collection of liquidity is never optimal.

It is not surprising to see a bank using its liquid investments to meet the liquidity demand in period one. However, what it does if it is faced with excess liquidity demand is more complex. Unlike Allen and Gale (2000), there is no complete liquidation pecking order. A bank chooses between long-term investments to be liquidated or its deposits to be withdrawn so to maximize the value of its investments, that is total welfare of its depositors. I study payments to consumers in cases when the entire network defaults, when just one bank defaults, and then proceed to the full solution of banks' equilibrium responses.

3.1. Liquidation Process

When the state is realized, banks pay off their liabilities using their resources. They may or may not be able to pay the unexpected costs arising from excess liquidity demand. If they are unable to meet the liabilities, they will default and liquidate all their assets and equally distribute them among all depositors. Eisenberg and Noe (2001) argue that there exists a transfer payment vector that clears the liabilities of all banks in a network. I investigate the market clearing payment vectors when default occurs.

Full Liquidation

Total welfare is minimum when the only market clearing payment vector is that of the case in which all banks default. In such cases, all banks liquidate all their assets in period one and equally distribute them among all depositors including the other bank, patient, and impatient consumers. Since total deposits a bank has made equals total deposits deposited in that bank, when both sides default the repayments of these deposits completely cancel each other out. So banks behave as if they have no cross-deposit channel connections whatsoever. I define C_{liq}^{all} as the withdrawal amount offered to consumers when the entire network defaults. C_{liq}^{all} has the following value which constitutes the value of liquid assets plus the liquidation value of iliquid assets.

$$C_{liq}^{all} = X + rY = \pi C_1 + r(1 - \pi C_1) \tag{2}$$

Note that $r < C_{liq}^{all} < 1$.

Partial liquidation

Full liquidation of the entire network is not the only possible outcome. It is also possible that one bank cannot meet the excess liquidity demand shock it is faced with and so defaults while the other bank is still able to manage its withdrawals, that is the other bank is still able to respect its contract. Without loss of generality, suppose bank A defaults and bank B survives. Bank B pays back C_1 units for each unit bank A has deposited in her bank. However, bank A is not able to respect his contract and, thus, provides C_{liq} units for each unit bank B has deposited in him. Since he liquidates all his assets and distributes them in period one, the amount he is able to pay back is the following,

$$C_{liq}(Z) = \frac{X + rY + ZC_1}{1+Z} = \frac{\pi C_1 + r(1-\pi C_1) + ZC_1}{1+Z}$$

$$C_{liq} = \frac{\pi C_1 + r(1-\pi C_1) + \omega C_1}{1+\omega} \qquad \text{if} \quad Z = \omega \tag{3}$$

Note that $r < C_{liq}^{all} < C_{liq}(Z) < C_1$. Bank A in this case can offer a higher final consumption to his depositors compared to when the entire financial system defaults, because he is paid C_1 for her deposits in other banks, but he pays C_{liq} for the other bank's deposits in his.

3.2. State S_3

In period one, bank B is faced with excess liquidity demand shock. Since her total liquid investments and the withdrawal value of her deposits in bank A worth $(\pi + \omega)C_1$ and total amount of liquidity she needs in period one is $(\pi + \omega + \epsilon)C_1$, she is not able to pay back all her liabilities without liquidating a fraction of her long-term investments.

Bank B withdraws all her deposits in bank A in period one. Otherwise, she has to inefficiently liquidate some of her long-term investments to compensate for her unwithdrawn deposits which can never be optimal. Bank A can also withdraw the entire or a fraction of his deposits in bank B in period one. Nonetheless, he does not need any additional liquidity in period one and such unnecessary withdrawal would result in less second period consumption level, incur more costs to bank B, and push it toward failure which will negatively affect bank A's asset values as well. Thus, bank A has no incentives to withdraw his deposits in bank B in period one.

Bank B should manage the excess liquidity demand she is faced. She has X units of liquid assets and ω units of cross-deposits that she fully withdraws. Since she needs $(\pi+\omega+\epsilon)C_1$, she liquidates the amount of $\frac{\epsilon C_1}{r}$ of her long-term investments. Thus, she will be left with $Y - \frac{\epsilon C_1}{r}$ untouched long-term investment in period one which yields $R(Y - \frac{\epsilon C_1}{r})$ in period two. The remaining depositors, a fraction $1-\pi-\omega-\epsilon$ of consumers in region B as well as bank A, will withdraw in period two, constituting a total fraction $1 - \pi - \epsilon$ of depositors.

Depositors are promised to receive C_2 units of consumption for each unit they have deposited in period zero. However, the costs incurred to bank B due to the excess liquidity demand shock, makes the optimal second period consumption infeasible. She, nonetheless, offers the the maximum feasible consumption for second period withdrawals. Patient depositors and bank A accept this offer as long as they are better off compared to those withdrawing in period one. If not, they will join the impatient consumers and withdraw early, causing a run and making bank B default.

If bank B liquidates too much of its long-term investments, there will be no enough untouched long-term investment to be harvested in period two that can provide an incentive compatible consumption for those withdrawing in period two. Therefore, a maximum level of nondiversifiable liquidity demand shock ϵ_1^{th} exists above which bank B's contract is not incentive compatible. If ϵ exceeds this threshold, then she has to default and liquidate all her investments in period one and distribute them among all depositors.

The threshold value ϵ_1^{th} is the maximum nondiversifiable liquidity shock that can be incentive compatibly managed without causing a run. Thus, if $\epsilon = \epsilon_1^{th}$, patient consumers will get the exact C_1 value in period two. Bank B equally distributes all the resources among all depositors. Thus, total withdrawals when $\epsilon = \epsilon_1^{th}$ holds equals to $(1-\pi-\epsilon_1^{th})C_1$ while total value of bank's assets equals to $R(Y - \frac{\epsilon_1^{th}C_1}{r})$. Therefore, the incentive compatibility condition is the following,

$$\begin{aligned} R(Y - \frac{\epsilon_1^{th}C_1}{r}) &= (1-\pi-\epsilon_1^{th})C_1 \\ (1-\pi)C_2 - \frac{\epsilon_1^{th}RC_1}{r} &= (1-\pi-\epsilon_1^{th})C_1 \\ \epsilon_1^{th} &= \frac{(1-\pi)(C_2-C_1)}{(\frac{R}{r}-1)C_1} \end{aligned} \tag{4}$$

Thus, if $\epsilon \leq \epsilon_1^{th}$, then bank B can offer an incentive compatible consumption schedule to the depositors without causing a run. On the other hand, if $\epsilon > \epsilon_1^{th}$, then bank B has to default and liquidate all her assets in period one and equally distribute them among all depositors.

Contagion

In this state and other unconventional states, if the bank paying the actual cost of premature liquidation of long-term investments, e.g. bank B in state S_3, does not default, i.e. $\epsilon \leq \epsilon_1^{th}$, the other bank connected to that does not default as well. The reason is that bank B can pay C_1 in period one to early withdrawals and a withdrawal amount greater than that in period two to late withdrawals. If there had been no excess liquidity demand shock, bank A connected to this bank would have paid C_1 to the early withdrawals he was faced out of his liquid investments and his withdrawals from bank B. Now that bank B is affected but survives, the withdrawal amount she pays is the same amount C_1. Thus, nothing is changed in terms of what all the banks can pay in period one with or without ϵ. In period two, however, bank B would have offered C_2 for late withdrawals. But she now offers a value less than C_2 and greater than C_1. Bank A's portfolio constitutes a portion of resources harvested from long-term investments yielding C_2 for each unit of deposits and the remaining resources withdrawn from bank B yielding $C_2 > C_2^B > C_1$. Thus, the average withdrawal value paid in second period is no less than C_2^B which itself is greater than C_1. Thus, bank A's actual consumption schedule is also incentive compatible and none of the banks default.

If $\epsilon > \epsilon_1^{th}$ holds so that bank B defaults, the payments depositors receive from bank B depends on the payments Bank A offers for bank B's deposits. If bank A is able to tolerate the loss incurred by Bank B's default, then it offers the same C_1 amount for all early withdrawals. However, Bank A's late payments should satisfy the incentive compatibility of patient depositors in his region preferring to withdraw late as well. Bank A receives the withdrawal value of C_{liq} given in equation (3). Bank A thus receives total value of ωC_{liq} for his deposits in bank B in period one. His liquid investments also worth πC_1. On the other hand, bank A's total liquidity demand is $(\pi - \omega)C_1$ and the withdrawal value of bank B deposits in bank A also worths ωC_1. Therefore, total liquidity demand is less than available liquid resources. and he does not need to prematurely liquidate any of his long-term investments. The remaining unused liquidity resources will be transferred to the second period.

In the second period, bank A can harvest his long-term investments and receive RY. He also transfers the unused liquidity of total value ωC_{liq} to period two. He serves the remaining patient consumers, a fraction of $(1 - \pi + \omega)$ of all consumers in region A, in period two. Thus, bank A's incentive compatibility constraint is as follows,

$$RY + \omega C_{liq} \geq (1 - \pi + \omega)C_1$$

$$(1 - \pi)(C_2 - C_1) \geq \omega(C_1 - C_{liq}) \tag{5}$$

$$\omega \leq \omega_1^{th} \tag{6}$$

A little algebra shows that the RHS of inequality (5) is increasing in ω. Thus, it is equivalent to inequality (6). If inequality (5) holds, bank A is indeed able to tolerate the costs of bank B's defaulting and survive, verifying my initial guess. If not, failure of bank B results in failure of bank A.

Therefore, if the $\epsilon > \epsilon_1^{th}$ holds so that bank B defaults, the payments depositors receive from bank B equals $C_{liq}^B = C_{liq}$ if inequality (5) holds and bank A does not default, and equals $C_{liq}^B = C_{liq}^{all}$ if inequality (5) does not hold and bank A also defaults.

3.3. State S_4

In this state bank A is faced with low realization of liquidity demand uncertainty and bank B is faced with the opposite. Bank A is also hit by a nondiversifiable liquidity demand shock. If bank B fully withdraws her deposits in bank A in period one, and uses her entire liquid investments, she can serve the liquidity demand from consumers in period one. However, if bank A also withdraws some of his deposits in bank B, she has no more available liquidity to serve that, unless she liquidates a fraction of her long-term investments at a cost. That is, there is no reason that bank B liquidates her long-term investments unless she has run out of other- less costly- liquidity sources. Therefore, bank B follows a liquidation pecking order.

Bank A, however, does not follow a liquidation pecking order. In fact, bank A takes into account the consequences of him withdrawing his deposits in bank B which can result in bank B's failure and, thus, can negatively affect bank A's asset values. As bank B follows a clear liquidation pecking order, the behavior of bank A can fully characterize equilibrium in this state. Bank A can strategically withdraw his entire deposits or a fraction of it in period one. Bank A chooses the actual level of withdrawals to maximize total welfare he can provide for his depositors. If it wasn't for ϵ, bank A wouldn't need to withdraw any amount in period one. However, he can strategically use this withdrawal option to pass the costs of excess liquidation to bank B. As explained earlier, it is never optimal for bank A to withdraw more than his actual excess liquidity demand ϵ. Nonetheless, it may be optimal to withdraw the entire ϵ or a fraction of it.

First, suppose bank A withdraws his total excess liquidity demand ϵ from his deposits in bank B in period one.[7] Bank B has to pay the period one value of bank A's withdrawals. This additional liquidity was not expected in period zero and her investment portfolio cannot costlessly provide this excess liquidity need. Since her liquid assets and her deposits in bank A are already depleted, she needs to liquidate some of her long-term investments to serve bank A. In other words, bank B should not only manage her high liquidity needs, but also must liquidate some of her long-term investments to serve the additional liquidity demand caused by bank A's passing his unexpected excess liquidity demand to bank B.

Bank B has πC_1 units of liquid assets, ωC_1 units of cross-deposits in bank A that she liquidates in period one, while she is faced with $(\pi + \omega + \epsilon)C_1$ units of liquidity demand in period one. Therefore, she has to liquidate $\frac{\epsilon C_1}{r}$ units of her long-term investments. Thus, bank B is left with $R(Y - \frac{\epsilon C_1}{r})$ units of resources in period two. She must pay a fraction $1 - \pi - \omega$ of consumers' and $\omega - \epsilon$ of the remaining Bank A's deposits, summing to a total fraction of $1 - \pi - \epsilon$. She must pay the patient depositors at least C_1 so that they prefer not to withdraw early. Therefore, the incentive compatibility condition is the following,

[7] I assume that all excess liquidity shock realizations are less than total cross-deposits ω unless otherwise stated. So that it is feasible for banks to withdraw that amount from their cross-deposits.

$$
\begin{aligned}
R(Y - \frac{\epsilon C_1}{r}) &\geq (1-\pi-\epsilon)C_1 \\
(1-\pi)C_2 - \frac{\epsilon R C_1}{r} &\geq (1-\pi-\epsilon)C_1 \\
\epsilon \leq \epsilon_1^{th} &= \frac{(1-\pi)(C_2 - C_1)}{(\frac{R}{r}-1)C_1}
\end{aligned}
$$

Which returns an excess liquidity demand threshold level similar to that in equation (4). Thus, the incentive compatibility constraint she faces is the same as that of state S_3. If $\epsilon \leq \epsilon_1^{th}$, then bank B survives. Otherwise, she defaults.

By passing all the costs of excess liquidity demand in period one to bank B, bank A only uses his liquid investments, i.e. πC_1, and a fraction of his deposits in bank B, i.e. ϵC_1, to pay off all the claims in period one. Nonetheless, the cost incurred to bank B will reduce the second period withdrawal amount she can offer to her depositors including bank A which subsequently affects the incentive compatibility constraint bank A is facing. This cost is, though, not large enough to break the incentive compatibility constraint of bank A as long as bank B doesn't default. The reason is that bank A should pay off a fraction $1-\pi+\omega-\epsilon$ of his depositors in period two. He harvests $RY = (1-\pi)C_2$ from his long-term investment and withdraws $(\omega - \epsilon)C_2^B$ of his deposits in bank B. Therefore, he offers the following second period consumption,

$$
C_2^A = \frac{(1-\pi)C_2 + (\omega-\epsilon)C_2^B}{1-\pi+\omega-\epsilon}
$$

Since $C_2^B \geq C_1$, it immediately follows that $C_2^A \geq C_1$. Therefore, if bank B survives, bank A will survive too.

Next, consider a liquidity demand shock to bank A that is greater than Bank B's maximum tolerable shock, i.e. $\epsilon > \epsilon_1^{th}$, and suppose bank A liquidates his deposits in bank B to serve the entire excess demand. Bank B cannot serve liquidity demand in period one and defaults. Thus, she liquidates all her assets and equally distributes them among her depositors in period one.

There are two possible equilibria regarding bank A. The first one is that bank A also defaults and, thus, the market clearing payment is that of when the entire network defaults, i.e. equation (2). The second equilibrium is when bank A does not default and is still able to offer an incentive compatible consumption schedule to his consumers. In the former, total value of bank B's withdrawals from bank A equals ωC_{liq}^{all}, and in the latter it equals ωC_1. Bank B then redistributes all the funds equally among all depositors (one unit of consumers and ω units of cross-deposits from bank A). Thus, the liquidation value for each unit of deposits she is able to offer equals C_{liq}^{all} in the former and C_{liq} in the latter.

I now consider bank A's problem given bank B's default and investigate under what conditions bank A survives. Bank A is faced with bank B withdrawing her entire deposits. His deposits in bank B are also mandatorily withdrawn and he receives the amount ωC_{liq} for his deposits. He must pay the excess liquidity demand he is faced with in period one. Depending on the relative value of what he receives from bank B and what he must pay to serve the excess liquidity demand, he may or may not need to liquidate some of his long-term investments. If bank A can pay off his liabilities in period one without liquidating his long-term investments, the magnitude of the shock should satisfy the following condition,

$$\begin{aligned}\pi C_1 + \omega C_{liq} &\geq (\pi - \omega + \epsilon)C_1 + \omega C_1 \\ \omega C_{liq} &\geq \epsilon C_1 \\ \epsilon \leq \epsilon_2^{th} &= \frac{\omega C_{liq}}{C_1} \end{aligned} \tag{7}$$

Thus, ϵ_2^{th} is the maximum excess liquidity demand that bank A can serve without liquidating his long-term investments when bank B defaults. Besides, bank A should still be able to offer an incentive compatible consumption schedule for patient consumers waiting until period two. In period two, bank A harvests his long-term investments and receives RY. He also transfers the unused liquidity $\omega C_{liq} - \epsilon C_1$ from period one. He must serve the remaining consumers which constitute a fraction $(1 - \pi + \omega - \epsilon)$ of all consumers depositing in A. Thus, the incentive compatibility constraint of bank A in the second period is the following,

$$RY + \omega C_{liq} - \epsilon C_1 \geq (1 - \pi + \omega - \epsilon)C_1$$

which simplifies to inequality (5). If it holds, bank A will survive after a shock with $\epsilon \leq \epsilon_2^{th}$ hitting bank A has made bank B default.

If $\epsilon > \epsilon_2^{th}$, i.e. bank A has to liquidate some of his long-term investments to serve period one withdrawal demands, he may still be able to offer an incentive compatible consumption schedule. He liquidates $\frac{\epsilon C_1 - \omega C_{liq}}{r}$ units of his long-term investments to meet the liquidity demand in period one. Thus, he will receive $R(Y - \frac{\epsilon C_1 - \omega C_{liq}}{r})$ units in period two. Similarly, he must serve the remaining consumers which constitute a fraction $(1 - \pi + \omega - \epsilon)$ of all consumers depositing in A. Thus, the incentive compatibility constraint of bank A in the second period is the following,

$$\begin{aligned} R[Y - \frac{\epsilon C_1 - \omega C_{liq}}{r}] &\geq (1 - \pi + \omega - \epsilon)C_1 \\ (1 - \pi)(C_2 - C_1) + \omega(\frac{R}{r}C_{liq} - C_1) &\geq \epsilon C_1(\frac{R}{r} - 1) \\ \frac{(1 - \pi)(C_2 - C_1) - \omega(C_1 - C_{liq}) + \omega(\frac{R}{r} - 1)C_{liq}}{C_1(\frac{R}{r} - 1)} &\geq \epsilon \\ \frac{(1 - \pi)(C_2 - C_1) - \omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)} + \frac{\omega C_{liq}}{C_1} &\geq \epsilon \end{aligned}$$

Thus,

$$\epsilon \leq \epsilon_3^{th} = \frac{(1-\pi)(C_2 - C_1) - \omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)} + \epsilon_2^{th} \tag{8}$$

$$\epsilon \leq \epsilon_3^{th} = \epsilon_1^{th} - \frac{\omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)} + \epsilon_2^{th} \tag{9}$$

Thus, when bank B defaults, if $\epsilon \leq \epsilon_3^{th}$, bank A can offer an incentive compatible consumption schedule to his consumers. In the following lemmas, I will investigate the circumstances under which the relative value of model parameters are specifiable. I will use the following results when solving for optimal bank A's strategy.

Lemma 1. *Condition $\epsilon_3^{th} > \epsilon_1^{th}$ holds. Furthermore, if inequality* (5) *holds, then $\epsilon_3^{th} \geq \epsilon_2^{th}$.*

Proof. Since $C_{liq} \geq r$ and $C_1 < R$, the following holds,

$$\frac{R}{r} C_{liq} \geq R > C_1 \tag{10}$$

Some algebra shows that inequalities in (10) result in inequality $\epsilon_2^{th} > \frac{\omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)}$ which itself is equivalent to inequality $\epsilon_3^{th} > \epsilon_1^{th}$. Furthermore, if inequality (5) holds, inequality (8) directly yields $\epsilon_3^{th} \geq \epsilon_2^{th}$. □

Lemma 2. *If the level of cross-deposits is less than a threshold, then $\epsilon_3^{th} \leq 2\epsilon_1^{th}$ holds.*

Proof.

$$\epsilon_3^{th} \leq 2\epsilon_1^{th}$$

$$\epsilon_1^{th} - \frac{\omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)} + \epsilon_2^{th} \leq 2\epsilon_1^{th}$$

$$\epsilon_2^{th} \leq \epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)} \tag{11}$$

$$\frac{\omega C_{liq}}{C_1} \leq \frac{(1-\pi)(C_2 - C_1) + \omega(C_1 - C_{liq})}{C_1(\frac{R}{r} - 1)}$$

$$(\frac{R}{r} - 1)\omega C_{liq} \leq (1-\pi)(C_2 - C_1) + \omega(C_1 - C_{liq})$$

$$\omega(\frac{R}{r} C_{liq} - C_1) \leq (1-\pi)(C_2 - C_1) \tag{12}$$

$$\omega \leq \omega_2^{th} \tag{13}$$

The LHS of inequality (12) is increasing in ω. Therefore a maximum level of cross-deposits exists above which the inequality (12) is violated. Therefore, inequality (12) is equivalent to inequality (13). [8] $\square$

Lemma 1 shows that if inequality (5) holds, then $\epsilon_2^{th} \leq \epsilon_3^{th}$. Thus, if bank A tolerates a shock that satisfies $\epsilon \leq \epsilon_2^{th}$, i.e. inequality (7) holds, then he will also tolerate a larger shock that satisfies $\epsilon_2^{th} < \epsilon \leq \epsilon_3^{th}$. In any case, when bank B defaults, bank A cannot tolerate a shock greater than ϵ_3^{th} and defaults as well.

Bank A's Strategic Withdrawal Decision

I have considered the equilibrium of economy and its stability supposing that bank A serves total excess demand he is faced with by withdrawing his deposits in bank B. The optimal withdrawal decision of bank A, however, is his parameter of choice. He may find it optimal to withdraw a different fraction of deposits than ϵ. Specifically, bank A may strategically decide not to withdraw the entire excess demand and tolerate this decision's direct costs so to keep bank B from defaulting , and avoid the extra costs incurred to him after bank B defaults. On the other hand, bank A may find it optimal to make bank B default and do so by withdrawing too much of his deposits in bank B. In this section, I will fully solve bank A's optimal liquidation decisions through the following four propositions.

Proposition 1. *If the shock hitting bank A is small enough, i.e. $\epsilon \leq \epsilon_1^{th}$, then it is always optimal for bank A to withdraw the entire excess liquidity demand shock he is faced with from his deposits in bank B. That is, it is optimal to pass all the direct liquidation costs to bank B.*

Proof. If bank A withdraws less than ϵ, he has to pay the remaining excess liquidity needs by liquidating some of his long-term investments and thus incurring a loss. Whereas if bank A withdraws the entire ϵ from bank B, he passes the direct withdrawal costs to her, and bank B must liquidate her long-term investments. That results in a reduction in second period consumptions she offers, but not her default. Since second period consumption is equal among all depositors, bank A incurs only a fraction of this loss and other depositors in bank B incur the remaining costs. Thus, the costs bank A incurs are less compared to not withdrawing the entire ϵ. As explained earlier, withdrawal of an amount more than ϵ is not optimal. Therefore, optimal withdrawal amount is ϵ itself. $\square$

Proposition 2. *If the shock hitting bank A is large enough, i.e. $\epsilon > \epsilon_1^{th}$, and inequality* (5) *does not hold, then it is weakly optimal for bank A to withdraw the exact amount ϵ_1^{th} of his deposits in bank B.*

[8] The relative value of ω_1^{th} and ω_2^{th} depends on model parameters. For instance, for economies with utility function of $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, the economy with parameter set of $(R, r, \pi, \omega, \sigma) = (1.5, 1, 0.1, 0.01, 4)$ features $\omega_1^{th} < \omega_2^{th}$, while the economy with that of $(1.5, 0.5, 0.1, 0.01, 4)$ features the opposite.

Proof. If bank A withdraws more than ϵ_1^{th} of his deposits in bank B, bank B will default. Since inequality (5) does not hold, bank A also defaults consequently and both banks can offer nothing better than the absolute minimum consumption level specified in equation (2) when the entire network defaults. Thus, withdrawing more than ϵ_1^{th} is not optimal. Using a similar logic to that in Proposition 1, withdrawing less than ϵ_1^{th} is also not optimal.

If bank A withdraws exactly ϵ_1^{th} of his deposits in bank B, and incurs the costs of remaining excess liquidity demand, he must liquidate the remaining excess liquidity demand, i.e. $C_1(\epsilon-\epsilon_1^{th})$, from his long-term investments. Thus, he can only harvest $R[y-\frac{C_1(\epsilon-\epsilon_1^{th})}{r}]$ units in period two. Besides, since he withdraws $\epsilon_1^{th}C_1$ in period one from bank B, bank B is able to offer exactly C_1 units of consumption for withdrawals in period two. Thus, he receives $C_1(\omega-\epsilon_1^{th})$ for his remaining deposits in bank B in period two. He must serve the fraction $(1-\pi+\omega-\epsilon)$ of consumers in period two. Thus, the following incentive compatibility condition of patient consumers should hold so that patient consumers in bank A wait until period two.

$$
\begin{aligned}
R[y-\frac{C_1(\epsilon-\epsilon_1^{th})}{r}]+C_1(\omega-\epsilon_1^{th}) &\geq (1-\pi+\omega-\epsilon)C_1 \\
(1-\pi)(C_2-C_1) &\geq C_1(\epsilon-\epsilon_1^{th})(\frac{R}{r}-1) \\
\epsilon &\leq \frac{(1-\pi)(C_2-C_1)}{C_1(\frac{R}{r}-1)}+\epsilon_1^{th}=2\epsilon_1^{th} \qquad (14)
\end{aligned}
$$

Thus, bank A can avoid the costs of defaulting of the entire network if he is hit by a shock $\epsilon \leq 2\epsilon_1^{th}$, in which he withdraws exactly ϵ_1^{th} from bank B. If shock is very large, i.e. $\epsilon > 2\epsilon_1^{th}$, bank A will also default independent of bank B's defaulting. Nonetheless, the output will be similar to bank A withdrawing ϵ_1^{th}. Thus, the weakly optimal withdrawal level is exactly ϵ_1^{th}. □

I now turn to the impacts of large liquidity shocks, i.e. $\epsilon > \epsilon_1^{th}$ in a ,seemingly , more natural setting in which bank A can tolerate bank B's failure in some cases. that is, inequality (5) holds. In this case, bank A can either withdraw all his deposits and make bank B default and have all his deposits liquidated in period one at a discounted rate, or choose not to withdraw all his deposits so that bank B survives and bank A is not faced with the contagion effects- at least in period one. The optimal action of bank A depends on the total welfare each option provides and those depend on the fundamental parameters of the model and the relative value of threshold parameters.

Proposition 3. *: If the shock hitting bank A is large enough, i.e. $\epsilon > \epsilon_1^{th}$, and inequality (5) holds, and if $\epsilon_3^{th} \leq 2\epsilon_1^{th}$, then it is weakly optimal for bank A to let bank B survive by withdrawing a fraction ϵ_1^{th} of his liquidity needs from bank B.*

Proof. If inequality (13) holds, then $\epsilon_3^{th} \leq 2\epsilon_1^{th}$. Considering the different relative values of ϵ_1^{th}, ϵ_2^{th} and the actual liquidity demand shock ϵ, I partition the entire state space to the following cases,

Case 1. $\epsilon_3^{th} \leq 2\epsilon_1^{th} < \epsilon$

As ϵ is greater than both ϵ_3^{th} and $2\epsilon_1^{th}$, regardless of bank A's choice to make bank B default or not, bank A also defaults. Thus, the only equilibrium is that both banks default and bank A is not better off by making bank B default.

Case 2. $\epsilon_3^{th} < \epsilon \leq 2\epsilon_1^{th}$

If bank A makes bank B default, since $\epsilon_3^{th} < \epsilon$, bank A also defaults. Whereas if bank A lets bank B survive by partially withdrawing his deposits in period one, since $\epsilon \leq 2\epsilon_1^{th}$, bank A also survives. Thus, letting bank B survive is strictly optimal for bank A.

Case 3. $\epsilon_2^{th} \leq \epsilon_1^{th} < \epsilon \leq \epsilon_3^{th} \leq 2\epsilon_1^{th}$ or $\epsilon_1^{th} \leq \epsilon_2^{th} < \epsilon \leq \epsilon_3^{th} \leq 2\epsilon_1^{th}$

Since $\epsilon \leq \epsilon_3^{th} \leq 2\epsilon_1^{th}$, regardless of bank A's decision of making bank B default or letting her survive bank A survives. Therefore, he is able to offer C_1 in first period. Thus, he chooses the strategy that results in higher second period consumption level. If bank A chooses to make bank B default, the following C_2^{A1} is the second period consumption level he is able to offer.

$$(1 - \pi + \omega - \epsilon)C_2^{A1} = R[y - \frac{\epsilon C_1 - \omega C_{liq}}{r}]$$

If he is to let bank B survive, he is best of by withdrawing exactly ϵ_1^{th} in period one and saving the remaining deposits for period two. Thus, the following C_2^{A2} is the second period consumption level he is able to offer.

$$(1 - \pi + \omega - \epsilon)C_2^{A2} = R[y - \frac{C_1(\epsilon - \epsilon_1^{th})}{r}] + C_1(\omega - \epsilon_1^{th})$$

Bank A will choose the second option if $C_2^{A1} \leq C_2^{A2}$ and the first option otherwise. Inequality $C_2^{A1} \leq C_2^{A2}$ is equivalent to,

$$
\begin{aligned}
R[y - \frac{\epsilon C_1 - \omega C_{liq}}{r}] &\leq R[y - \frac{C_1(\epsilon - \epsilon_1^{th})}{r}] + C_1(\omega - \epsilon_1^{th}) \\
\frac{R}{r}[C_1(\epsilon - \epsilon_1^{th}) - (\epsilon C_1 - \omega C_{liq})] &\leq C_1(\omega - \epsilon_1^{th}) \\
\frac{R}{r}[\omega C_{liq} - C_1 \epsilon_1^{th}] &\leq C_1(\omega - \epsilon_1^{th}) \\
\omega[\frac{R}{r} C_{liq} - C_1] &\leq C_1 \epsilon_1^{th}[\frac{R}{r} - 1] \\
\omega[\frac{R}{r} C_{liq} - C_1] &\leq (1 - \pi)(C_2 - C_1)
\end{aligned}
$$

which is similar to inequality (12). Therefore, the weakly optimal decision of bank A is to let bank B survive.

Case 4. $\epsilon_1^{th} < \epsilon \leq \epsilon_2^{th} \leq \epsilon_3^{th} \leq 2\epsilon_1^{th}$

Since $\epsilon \leq \epsilon_3^{th} \leq 2\epsilon_1^{th}$, regardless of bank A's decision of making bank B default or letting her survive bank A survives. Therefore, he is able to offer C_1 in the first period. He chooses the strategy that results in higher second period consumption level. If bank A chooses to make bank B fail, the following C_2^{A1} is the second period consumption level he is able to offer.

$$(1 - \pi + \omega - \epsilon) C_2^{A1} = Ry + \omega C_{liq} - \epsilon C_1$$

If he is to let bank B survive, he is best of by withdrawing exactly ϵ_1^{th} in period one and saving the remaining deposits for period two. Thus, the following C_2^{A2} is the second period consumption level he is able to offer.

$$(1 - \pi + \omega - \epsilon) C_2^{A2} = R[y - \frac{C_1(\epsilon - \epsilon_1^{th})}{r}] + C_1(\omega - \epsilon_1^{th})$$

Bank A will choose the second option if $C_2^{A1} \leq C_2^{A2}$ and the first option otherwise. Condition $C_2^{A1} \leq C_2^{A2}$ is equivalent to,

$$
\begin{aligned}
Ry + \omega C_{liq} - \epsilon C_1 &\leq R[y - \frac{C_1(\epsilon - \epsilon_1^{th})}{r}] + C_1(\omega - \epsilon_1^{th}) \\
C_1(\epsilon - \epsilon_1^{th})[\frac{R}{r} - 1] &\leq \omega(C_1 - C_{liq}) \\
\epsilon &\leq \epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1[\frac{R}{r} - 1]} \qquad (15)
\end{aligned}
$$

The condition $\epsilon_3^{th} \leq 2\epsilon_1^{th}$ is equivalent to $\epsilon_2^{th} \leq \epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1(\frac{R}{r}-1)}$ (inequality (11)). Since $\epsilon \leq \epsilon_2^{th}$ inequality (15) always holds. Therefore, Bank A's optimal decision is to let bank B survive.

Having considered all the possible cases, Bank A is either indifferent between the two options or strictly prefers letting bank B survive. Therefore, letting bank B survive is bank A's weakly optimal strategy. □

Proposition 4. *If the shock hitting bank A is large enough, i.e.* $\epsilon > \epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1[\frac{R}{r}-1]}$, *(not large enough, i.e.* $\epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1[\frac{R}{r}-1]} \geq \epsilon > \epsilon_1^{th}$*) and inequality* (5) *holds, and if* $\epsilon_3^{th} > 2\epsilon_1^{th}$*, then it is weakly optimal for bank A to make bank B default (let bank B survive by withdrawing a fraction* ϵ_1^{th} *of his liquidity needs from bank B).*

Proof. If inequality (12) does not hold, then $\epsilon_3^{th} > 2\epsilon_1^{th}$ holds. It also immediately follows by plugging in this inequality into inequality (9) that $\epsilon_2^{th} > \epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1(\frac{R}{r}-1)} > \epsilon_1^{th}$. Therefore, I partition the entire state space regarding the relative values of threshold parameters and the actual excess liquidity demand ϵ to the following cases,

Case 1. $2\epsilon_1^{th} < \epsilon_3^{th} < \epsilon$

ϵ is large enough that regardless of bank A's decision both banks will default. So he does not have a strict preference on his withdrawal decisions in this case.

Case 2. $2\epsilon_1^{th} < \epsilon \leq \epsilon_3^{th}$

If bank A chooses to let bank B survive, he will default. However, if bank A makes bank B fail, he will survive. Therefore, bank A strictly prefers the latter choice.

Case 3. $\epsilon_1^{th} < \epsilon_2^{th} < \epsilon \leq 2\epsilon_1^{th} < \epsilon_3^{th}$

In this case both bank A's decisions result in his survival. Thus, he compares second period consumption level he can offer in each strategy. The calculations are similar to those of case 3 of proposition 3, and bank A's choice depends on the direction of inequality 12. Since inequality (12) does not hold, the optimal decision in this case is to make bank B default.

Case 4. $\epsilon_1^{th} < \epsilon \leq \epsilon_2^{th} \leq 2\epsilon_1^{th} < \epsilon_3^{th}$ or $\epsilon_1^{th} < \epsilon \leq 2\epsilon_1^{th} \leq \epsilon_2^{th} < \epsilon_3^{th}$

Similar to case 4 of proposition 3, both bank A's decisions result in his survival. So he compares second period consumption level he can offer in each strategy. The calculations are similar and show that making bank B default is bank A's optimal strategy if $\epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1[\frac{R}{r}-1]} < \epsilon \leq \epsilon_2^{th}$ and letting bank B survive is his optimal strategy if $\epsilon_1^{th} < \epsilon \leq \epsilon_1^{th} + \frac{\omega(C_1 - C_{liq})}{C_1[\frac{R}{r}-1]}$.

□

4. Discussion

My model studies a financial network in which banks are subject to a regional liquidity demand uncertainty- without aggregate effects- and excess liquidity demand shocks. Similar to Allen and Gale (2000), I study stability of the financial network when excess liquidity demand shock hit the economy. However, Allen and Gale (2000) assume that banks are hit by either liquidity demand uncertainty or liquidity demand shocks, whereas, I study a network in which banks are heterogeneous in terms of their liquidity needs when are hit by an excess liquidity demand shock and consequently generate heterogeneous responses. I show when a bank with low liquidity needs- which I call the "safe" bank- is hit by an excess liquidity demand shock, it passes the incurred direct costs of providing liquidity to the bank with high liquidity needs- which I call the "unsafe" bank. An strategic action that benefits first bank only through second bank's loss.

Furthermore, unlike Allen and Gale (2000), there is no complete pecking order in banks' liquidation decisions. They choose which investments to liquidate so to maximize their depositors' total welfare. If the excess liquidity shock is small enough, the safe bank chooses to pass the costs in their entirety to the unsafe bank. However, if the shock is large enough such that passing the costs to the unsafe bank will make it default, the safe bank may choose not to do so to avoid the spillover costs of unsafe bank's failure. Whether the safe bank chooses to control the damage to the unsafe bank, depends on the fundamental parameters of the economy as well as the magnitude of cross-deposits banks have made in each other. In some cases the safe bank chooses to pass the liquidity demand such that the unsafe bank is pushed to the brink of failure but does not default, and incur the cost of serving the remaining excess liquidity demand by itself. An action that- consistent with that of Leitner (2005)- can be considered as a private bailout to let the unsafe bank survive. However, in some other cases the safe bank chooses to make the unsafe bank default by withdrawing too much of its deposits in the unsafe bank. Note that as shown in inequality (13), the selection between the two cases depends on the level of cross-deposits banks make in each other. In some cases, if banks are too connected, i.e. $Z > \omega_2^{th}$, then it is optimal to make the unsafe bank default. Stiglitz (2010) argues that complete financial networks are not necessarily optimal and a network structure consisting of several regions that are well-connected within each region, but have limited connections between regions is more stable. I show that not only the presence of financial links but the magnitude of them, if is greater than a threshold, negatively affects network stability and triggers contagion across different regions. Thus, a relevant policy recommendation to stabilize the financial system is to control the extent of connections intermediaries form with other ones.

It is also important to note that contagion does not only depend on model parameters as in Allen and Gale (2000), but also on whom is hit by liquidity demand shock. Furthermore, regardless of the bank being hit by the liquidity shock, "unsafe" banks pay the direct costs of liquidations[9] and are taken to the brink of failure, if they are not forced to default.

4.1. The Impacts of Higher Cross-deposit Levels

As discussed in subsection 2.3, all the contracts with cross-deposit levels higher than ω, i.e. the level that is necessary to fully insure banks against liquidity demand uncertainty, yield the first-best consumption schedule in conventional states. In conventional states any excess liquidation of cross-deposits by one side is completely retaliated by the other side, and the Nash equilibrium is that both sides withdraw these excess cross-deposits, i.e. $Z - \omega$, in period one. Thus, cross-deposit levels higher than ω are neutral in conventional states. However, analysis of unconventional states is more complex. Suppose a financial network similar to the one described in section (3) but with a higher cross-deposit level ($Z > \omega$). First observe that when the entire network defaults, the payment vector banks are able to offer is similar to that for $Z = \omega$, and when one side defaults and the other does not, the payment amount the failing bank is able to offer($C_{liq}(Z)$) is increasing in Z. Thus, consumers in the failing bank's region benefit from a higher cross-deposit level, and consumers in the other region incur a loss. It is the case because the other bank respects its contract and pays C_1 for each unit of deposits the failing bank has made in it while being paid $C_{liq} < C_1$ for each unit of deposits it has made in the failing bank. More specifically, the net outflow from the surviving bank to the failing bank is $Z(C_1 - C_{liq})$, and it is, not surprizingly, increasing in Z. Thus, the cost of one bank's default is higher for the other bank when Z is increased.

Consider state S_3. The maximum amount of excess liquidity shock a bank can tolerate, does not depend on Z. In fact, any excess withdrawals from bank B will be retaliated by bank A in the same manner as in conventional states. Thus, bank B has to serve the excess liquidity demand by liquidating her long-term assets. Equivalently, $\epsilon_1^{th}(Z) = \epsilon_1^{th}$. Nonetheless, the incentive compatibility condition of bank A when bank B defaults, i.e. inequality (5) when $Z = \omega$, changes to the following condition,

$$(1-\pi)(C_2 - C_1) \geq Z(C_1 - C_{liq}(Z)) \tag{16}$$

The RHS of this inequality is increasing in Z, while the LHS is constant. Therefore, as Z is increased, it gets more difficult for bank A to survive when

[9] The safe bank still incurs the indirect costs of the other bank's liquidation through network spillover costs.

bank B defaults. In other words, the financial system is now more prone to contagion than it was when $Z = \omega$.

I define $\epsilon_2^{th}(Z)$ as the maximum magnitude of a shock hitting bank A that he can tolerate without liquidating any of his long-term assets when he makes bank B default and $\epsilon_3^{th}(Z)$ as the maximum magnitude of a shock hitting bank A that he can tolerate after bank A makes bank B default.[10] Some algebra yield the following results,

$$\epsilon_2^{th}(Z) = \omega - \frac{Z(C_1 - C_{liq})}{C_1}$$
$$\epsilon_3^{th}(Z) = \epsilon_1^{th} + \epsilon_2^{th}(Z) - \frac{Z(C_1 - C_{liq})}{(\frac{R}{r} - 1)C_1}$$

It can be easily shown that these threshold values are both smaller than their respective counterparts when $Z = \omega$ and are also decreasing in Z.

Next, consider state S_4 when $Z > \omega$. If Z is increased, bank A incurs more loss when bank B defaults. On the other hand, if bank B survives, the actual level of Z does not have welfare impacts on consumers in bank A's region. Therefore, as Z is increased, bank A gets more resilient to make bank B default and more frequently engages in private bailouts to bank B. I will describe how bank A behaves and how my results change when $Z > \omega$ in the following.

Given the conditions necessary for proposition 1, no bank defaults. Thus, the final allocation when $Z > \omega$ will be similar to that of $Z = \omega$. The only exception is that both banks withdraw excess cross-deposits, i.e. $Z - \omega$, in period one. So the excess cross-deposits cancel each other out.

If the conditions of proposition 2 hold for an economy when $Z = \omega$, they will also hold for the same economy but with $Z > \omega$. Because the incentive compatibility condition of bank A when bank B defaults becomes more difficult to satisfy as Z is increased. Thus, if it does not hold for $Z = \omega$, it will not hold for any $Z > \omega$.

Propositions 3 needs an extra assumption to begin with. I need to assume that incentive compatibility condition of bank A when bank B defaults is still satisfied for larger Z. Supposing that is the case, if the conditions of proposition 3 hold for an economy when $Z = \omega$, they will also hold for the same economy but with $Z > \omega$. Because, $\epsilon_3^{th}(Z) < \epsilon_3^{th}$ and if $\epsilon_3^{th} \leq 2\epsilon_1^{th}$ it follows immediately that $\epsilon_3^{th}(Z) \leq 2\epsilon_1^{th}$. The condition $\epsilon_3^{th} > 2\epsilon_1^{th}$ of proposition 4, on the other side, become harder to satisfy for the same reason. Thus, the circumstances under which bank A prefers to make bank B default, become less common.

The Most Stable Optimal Cross-deposit Level

In my model one bank has more ex-post liquidation decisions to make. Thus, two sides do not have equal ex-post negotiation powers. If unconventional states

[10]These definitions respectively simplify to those of of ϵ_2^{th} and ϵ_3^{th} when $Z = \omega$.

would happen with a nonzero probability, that would bring up the hold-up problem in banks' ex-ante behavior which is not the case in the economies I study. Nonetheless, there exist some investment portfolios that provide the first-best consumption schedule in states happening with nonzero probability, but behave differently in unconventional states. If the level of cross-deposits is large enough so that inequality (16) is violated, then failure of one bank results in contagion and failure of the other bank. On the other hand, as Z is increased, the loss incurred to one bank given the default of the other bank increases, resulting in more private bailouts and less actual failures of the unsafe bank. Thus , one can say higher cross-deposit levels have a convoluted effect. Given the possible choices of Z, none of investment portfolios with cross-deposits less than a threshold, i.e. $Z < \omega$, can provide the first-best consumption schedule. Additionally, all the levels that violate inequality (16) make the financial system too fragile. As the cross-deposit level is increased between the two thresholds, it does not invert the sign of inequality (16) and it makes the banks more cooperative and lowers the likelihood of a bank defaulting. Thus, the deposit level that solves inequality (16) , i.e. $Z = \omega_1^{th}$, is the optimal level given that inequality 5 holds.

5. Conclusions

In a financial network with heterogeneous intermediaries, banks with low liquidity needs pass the unexpected liquidation costs to banks with high liquidity needs. There is no complete pecking order, and the decision of what investment to liquidate is an endogenous, and nontrivial, decision of banks. Safe banks may choose to engage in private bailouts to let the unsafe banks survive and, thus, control the indirect costs of liquidation feeding bank to their asset values. Furthermore, only moderate levels of cross-deposits are optimal. Low levels do not yield the first-best consumption schedule, and high levels increase fragility. In some economies, the most stable optimal contract specifies a a cross-deposit level that is larger than the minimum level needed to fully insure banks against liquidity demand uncertainty.

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