

Supporting Materials for:
Two Monetary Models with Alternating Markets

Gabriele Camera	YiLi Chien
Chapman University	Federal Reserve Bank of St. Louis
University of Basel	

1 Optimal choices in the CIA model

On date t , given history S^t , the constraint of the firm is

$$F(h_t^F(S^t)) = c_{1t}^F(S^t) + c_{2t}^F(S^t) \quad (1)$$

where c_{1t}^F and c_{2t}^F denote cash and credit goods, p_{jt} is the nominal spot price of good $j = 1, 2$, and w_t is the nominal spot wage on t . Nominal profits (net dollar inflows) are distributed as dividends in the afternoon, and on the morning of t are

$$p_{1t}(S^t)c_{1t}^F(S^t) + p_{2t}(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t). \quad (2)$$

Since the firm sells for cash *and* for credit, payments accrue as follows: in the morning, it receives cash payments for cash-goods sales, and in the afternoon it receives payments for the morning's credit sales. Let $q_t(S^t)$ denote the date-0 price of a claim to one dollar delivered in the afternoon of t , contingent on S^t (= state-contingent nominal bond). The firm's date-0 profit-maximization problem is: given state-contingent prices $q_t(S^t)$, choose sequences of output and labor $(c_{1t}^F(S^t), c_{2t}^F(S^t), h_t^F(S^t))$ to solve

$$\text{Maximize: } \sum_{t=0}^{\infty} \int q_t(S^t) \{p_{1t}(S^t)c_{1t}^F(S^t) + p_{2t}(S^t)c_{2t}^F(S^t) - w_t(S^t)h_t^F(S^t)\} dS^t \quad (3)$$

$$\text{subject to: } c_{1t}^F(S^t) + c_{2t}^F(S^t) = F(h_t^F(S^t)).$$

Substituting for $c_{1t}^F(S^t)$ from the constraint, the FOCs for all t, S^t are

$$\begin{aligned} h_t^F(S^t) : \quad & p_{1t}(S^t)F'(h_t^F(S^t)) - w_t(S^t) = 0 \\ c_{2t}^F(S^t) : \quad & p_{1t}(S^t) - p_{2t}(S^t) = 0. \end{aligned}$$

Consequently, for all t, S^t we have $p_{1t}(S^t) = p_{2t}(S^t) = p_t(S^t)$ and

$$p_t(S^t)F'(h_t^F(S^t)) = w_t(S^t). \quad (4)$$

An agent who contracts on date 0 maximizes the expected utility

$$\sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f^t(S^t) dS^t$$

where we assume U is a real-valued function, twice continuously differentiable in each argument, strictly increasing in c_j , decreasing in h , and concave. Maximization is subject to two constraints. One is the cash in advance constraint

$$p_{1t}(S^t)c_{1t}(S^t) \leq M_t(S^{t-1}) \quad \text{for all } t \text{ and } S^t,$$

where $M_t(S^{t-1})$ are money balances held at the start of t , brought in from the afternoon of $t-1$, when the shock s_t was not yet realized. Given this uncertainty, money may be held to conduct transactions and for precautionary reasons.

The other constraint is the date-0 nominal intertemporal budget constraint:

$$\begin{aligned} \sum_{t=0}^{\infty} \int \left\{ q_t(S^t) \left[p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_t(S^t)h_t(S^t) - M_t(S^{t-1}) \right. \right. \\ \left. \left. + M_{t+1}(S^t) - \Theta_t \right] \right\} dS^t \leq \Pi + \bar{M}. \end{aligned}$$

The date-0 sources of funds are \bar{M} initial money holdings (=initial liabilities of the central bank) and the firm's nominal value Π . The left hand side is the date-0 present value of net expenditure. It is calculated by considering the price of money delivered

in the afternoon of t , $q_t(S^t)$. There are two elements:

1. Morning net expenditure: $w_t(S^t)h_t(S^t)$ wages earned, paid in the afternoon; $M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)$ unspent balances available in the afternoon; $p_{2t}(S^t)c_{2t}(S^t)$ purchases of credit goods settled in the afternoon. These funds are available in the afternoon of t , where the date-0 value of one dollar is $q_t(S^t)$.
2. Afternoon net expenditures: the agent receives Θ_t transfers and exits the period holding $M_{t+1}(S^t)$ money balances, so net expenditure is $M_{t+1}(S^t) - \Theta_t$, with date-0 value $q_t(S^t)$.

Given that values can be history-dependent, we integrate over S^t .

Agents choose sequences of state-contingent consumption, labor and money holdings $c_{1t}(S^t)$, $c_{2t}(S^t)$, $h_t(S^t)$, and $M_{t+1}(S^t)$ to maximize the Lagrangian:

$$\begin{aligned}
L := & \sum_{t=0}^{\infty} \beta^t \int U(c_{1t}(S^t), c_{2t}(S^t), h_t(S^t)) f^t(S^t) dS^t + \lambda(\Pi + \bar{M}) \\
& - \lambda \sum_{t=0}^{\infty} \int \{q_t(S^t)[p_{1t}(S^t)c_{1t}(S^t) + p_{2t}(S^t)c_{2t}(S^t) - w_t(S^t)h_t(S^t) \\
& - M_t(S^{t-1}) + M_{t+1}(S^t) - \Theta_t]\} dS^t \\
& + \sum_{t=0}^{\infty} \int \mu_t(S^t)[M_t(S^{t-1}) - p_{1t}(S^t)c_{1t}(S^t)] dS^t,
\end{aligned} \tag{5}$$

where $\mu_t(S^t)$ is the Kuhn-Tucker multiplier on the cash constraint on t , given S^t .

Omitting the arguments from U and f where understood, in an interior optimum the FOCs for all t and S^t are:

$$\begin{aligned}
c_{1t}(S^t) : & \quad \beta^t U_1 f^t(S^t) - \lambda p_{1t}(S^t) q_t(S^t) - \mu_t(S^t) p_{1t}(S^t) = 0 \\
& \quad p_{1t}(S^t) c_{1t}(S^t) \leq M_t(S^{t-1}) \\
c_{2t}(S^t) : & \quad \beta^t U_2 f^t(S^t) - \lambda p_{2t}(S^t) q_t(S^t) = 0 \\
h_t(S^t) : & \quad \beta^t U_3 f^t(S^t) + \lambda w_t(S^t) q_t(S^t) = 0 \\
M_{t+1}(S^t) : & \quad -\lambda q_t(S^t) + \lambda \int q_{t+1}(S^{t+1}) ds_{t+1} + \int \mu_{t+1}(S^{t+1}) ds_{t+1} = 0.
\end{aligned} \tag{6}$$

Given $p_{2t}(S^t) = p_{1t}(S^t) = p(S^t)$ and (4) we get

$$\frac{U_3}{U_2} = F'(h_t(S^t); S^t) \quad \text{for all } t, S^t$$

$$\frac{U_1}{U_2} = \frac{\lambda q_t(S^t) + \mu_t(S^t)}{\lambda q_t(S^t)} \quad \text{for all } t, S^t. \quad (7)$$

2 The price distortion in the LW model

Under bargaining, $\frac{u'_1(c_1)}{z'(c_1; \theta)}$ is the marginal benefit from spending a dollar. This ratio becomes $\frac{u'_1(c_1)}{p_1/p_2}$, with $\frac{p_1}{p_2} = \eta'(c_1)$, when $\theta = 1$. To see this, note that if $\theta = 1$, then $z' = \eta'$. If $\theta < 1$ we have $z' > \eta'$. Indeed, $u'_1 \geq \eta'$; hence, $\theta u'_1 + (1 - \theta)\eta' < u'_1$. From the definition of $z(c_1; \theta)$ we have $z' = \frac{u'_1}{\theta u'_1 + (1 - \theta)\eta'} \eta' + A$ where $A > 0$.

The Figure plots $\psi(c_1, \theta)$ to illustrate how Nash bargaining distorts prices, relative to competitive pricing, depending on the buyer's bargaining power θ , and the rate of inflation. As θ approaches one, the price distortion vanishes for any rate of inflation, and the Nash bargaining price distortion vanishes.

3 Proof of Lemma 1

Consider an equilibrium with history-independent prices $p_{1t}(S^t) = p_{1t}$ and $w_{1t}(S^t) = w_{1t}$, as in (LW, 2005).⁶ To prove the first part of the Lemma let $s_t^i = 1$ and $\mu_t(S^t) = 0$. From the first and third expressions in (12) we have

$$\beta^t u'_1(c_{1t}(S^t)) = \lambda p_{1t} q_t = \lambda w_{1t} q_t = \beta^t \eta'(h_{1t}(S^t)), \quad \text{for all } t, S^t,$$

From market clearing $h_{1t}^F(S^t) = \delta h_{1t}(S^t) = \delta c_{1t}(S^t) = c_{1t}^F(S^t)$.⁷ Hence, $\frac{u'_1(c_{1t}(S^t))}{\eta'(c_{1t}(S^t))} = 1$ for all t, S^t . That is $c_{1t}(S^t) = c_1$ for all t and all agents i such that $s_t^i = 1$.

To prove the second part of the Lemma let $s_t^i = 1$ and $\mu_t(S^t) > 0$. Update by one

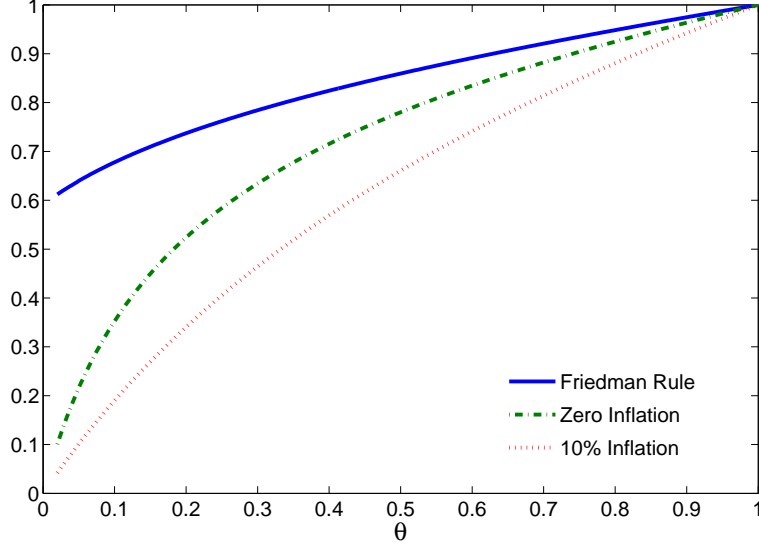


Figure 1: Illustrating the bargaining price distortion using $\psi(c_1, \theta)$

Notes: The three curves correspond to $\psi(c_1; \theta)$ assuming—as in the calibration in (LW, 2005)—that $\eta' = 1$, $u_1(c_1) = \frac{(c_1+b)^{1-a}-b^{1-a}}{1-a}$, $a = 0.3$, $b = 0$, $\delta = 0.5$ and $r = 1.04\gamma - 1$, with $\gamma = \beta$ (=Friedman rule), $\gamma = 1$ (=zero inflation) and $\gamma = 1.1$ (=10% inflation).

period the first expression in the FOCs (12) to get

$$\frac{\beta^{t+1}}{p_{1,t+1}} u'_1(c_{1,t+1}(S^{t+1})) f(s_{t+1}) f^t(S^t) = \lambda q_{t+1} f(s_{t+1}) f^t(S^t) + \mu_{t+1}(S^{t+1}), \quad \text{if } s_{t+1}^i = 1$$

where we substituted $f^{t+1}(S^{t+1}) = f(s_{t+1}) f^t(S^t)$. Now substitute $c_{1,t+1}(S^{t+1}) = \frac{M_{t+1}(S^{t+1})}{p_{1,t+1}}$ since $\mu_{t+1}(S^{t+1}) > 0$. The expression above has the status of an equality *only if* $s_{t+1}^i = 1$. In that case, we can integrate both sides with respect to s_{t+1} ,

conditional on $s_{t+1}^i = 1$. For the left-hand-side we get

$$\begin{aligned}
& \frac{\beta^{t+1}}{p_{1,t+1}} \int \mathbf{1}_{\{s_{t+1}^i=1\}} u'_1(c_{1,t+1}(S^{t+1})) f(s_{t+1}) f^t(S^t) d_{s_{t+1}} \\
&= \frac{\beta^{t+1}}{p_{1,t+1}} u'_1\left(\frac{M_{t+1}(S^t)}{p_{1,t+1}}\right) \int \mathbf{1}_{\{s_{t+1}^i=1\}} f(s_{t+1}) f^t(S^t) d_{s_{t+1}} \\
&= \frac{\beta^{t+1}}{p_{1,t+1}} u'_1\left(\frac{M_{t+1}(S^t)}{p_{1,t+1}}\right) f^t(S^t) \int \mathbf{1}_{\{s_{t+1}^i=1\}} f(s_{t+1}) d_{s_{t+1}} \\
&= \frac{\beta^{t+1}}{p_{1,t+1}} u'_1\left(\frac{M_{t+1}(S^t)}{p_{1,t+1}}\right) f^t(S^t) \delta
\end{aligned} \tag{8}$$

For the right-hand-side we get

$$\begin{aligned}
& \int \mathbf{1}_{\{s_{t+1}^i=1\}} [\lambda q_{t+1} f(s_{t+1}) f^t(S^t) + \mu_{t+1}(S^{t+1})] d_{s_{t+1}} \\
&= \lambda q_{t+1} f^t(S^t) + \int \mu_{t+1}(S^{t+1}) d_{s_{t+1}} - \Phi = \lambda q_t f^t(S^t) - \Phi,
\end{aligned} \tag{9}$$

where the last step follows from the last line in (12) and

$$\begin{aligned}
\Phi &:= \int \mathbf{1}_{\{s_{t+1}^i=0\}} [\lambda q_{t+1} f(s_{t+1}) f^t(S^t) + \mu_{t+1}(S^{t+1})] d_{s_{t+1}} \\
&= \int \mathbf{1}_{\{s_{t+1}^i=0\}} [\lambda q_{t+1} f(s_{t+1}) f^t(S^t)] d_{s_{t+1}}, \quad \text{since } \mu_{t+1}(S^{t+1}) = 0 \text{ when } s_{t+1}^i = 0 \\
&= \lambda q_{t+1} f^t(S^t) \int \mathbf{1}_{\{s_{t+1}^i=0\}} f(s_{t+1}) d_{s_{t+1}} \\
&= \lambda q_{t+1} f^t(S^t) (1 - \delta), \quad \text{since } \int \mathbf{1}_{\{s_{t+1}^i=0\}} f(s_{t+1}) d_{s_{t+1}} = 1 - \delta \\
&= \beta^{t+1} \frac{u'_2(c_{2,t+1})}{p_{2,t+1}} f^t(S^t) (1 - \delta), \quad \text{from (12)}.
\end{aligned}$$

Equating the expectations of both sides from (8) and (9) we have

$$\frac{\beta^{t+1}}{p_{1,t+1}} u'_1\left(\frac{M_{t+1}(S^t)}{p_{1,t+1}}\right) \delta = \lambda q_t - \frac{\Phi}{f^t(S^t)}$$

Substituting Φ in the equation above we get

$$\frac{\beta^{t+1}}{p_{1,t+1}} u'_1\left(\frac{M_{t+1}(S^t)}{p_{1,t+1}}\right) \delta = \lambda q_t - \frac{\beta^{t+1} u'_2(c_{2,t+1})}{p_{2,t+1}} (1 - \delta),$$

or equivalently, since $u_2'(c_{2,t+1}) = 1$ for all $t + 1$ and S^{t+1} , we have

$$\beta^{t+1} \left[u_1' \left(\frac{M_{t+1}(S^t)}{p_{1,t+1}} \right) \frac{\delta}{p_{1,t+1}} + \frac{1 - \delta}{p_{2,t+1}} \right] = \lambda_{q_t}.$$

This implies that if $s_{t+1}^i = 1$, then $c_{1,t+1}(S^{t+1}) = \frac{M_{t+1}(S^t)}{p_{1,t+1}} = \frac{M_{t+1}}{p_{1,t+1}} = c_{1,t+1}$ for all t and S^t and for all agents i , because q_t is independent of S^t . The distribution of money is degenerate because there are no wealth effects due to the linear disutility from producing credit goods. Agents equally reach the same cash holdings by adjusting their labor supply h_2^i . By market clearing, $h_{2t}^F = \int h_{2t}^i di = c_{2t}$ where h_{2t}^i satisfies the agents' budget constraint.

Now substitute $\lambda_{q_t} = \frac{\beta^t u_2'(c_{2t})}{p_{2t}} = \frac{\beta^t}{p_{2t}}$ from (12) and write the equation above as (13). Finally, from the firm's problem, we have $\eta'(h_{1t}) = \frac{w_{1t}}{w_{2t}} = \frac{p_{1t}}{p_{2t}}$. ■

4 Comparing notations in LW and our model

In LW, $U(X)$ is the utility received from consuming X CM goods ($u_2(c_2)$ in our notation). The technology to produce CM goods is linear and the disutility from labor is linear. In the DM, a portion $\alpha\sigma$ (δ in our notation) of agents desires to consume (but cannot produce) and an identical portion can produce but does not consume; $u(q)$ is the utility received from consuming q DM goods ($u_1(c_1)$ in our notation); c is the disutility from labor in the DM (η in our notation); the nominal price is $\frac{d}{q}$ per unit of consumption (p_1 in our notation); the real price is $\frac{\phi d}{q}$, where ϕ is $\frac{1}{p_2}$ in our notation. With binding cash constraints $d = M$ and $\frac{\phi M}{q}$ where M is the agent's money holdings. We also have $\phi M \equiv z(q)$ where $0 < \theta \leq 1$ is the buyer's bargaining power. The nominal interest rate is i (r in our notation).

5 Details about the quantitative exercise

Preferences specification: Preferences over goods are defined by

$$u_1(c_1) = \frac{(c_1 + b)^{1-a} - b^{1-a}}{1-a} \quad \text{and} \quad u_2(c_2) = B \log c_2,$$

for some $a > 0$, $b \in (0, 1)$ and $B > 0$. Consumption c_2 satisfies (6), labor disutility satisfies $\eta' = 1$, so c_1 satisfies

$$\frac{\gamma}{\beta} - 1 = \delta[\tau u_1'(c_1) - 1]. \tag{10}$$

The welfare cost of inflation: Define ex-ante welfare

$$\mathcal{W}_\gamma := u_2(c_2) - c_2 + \delta[u_1(c_1(\gamma)) - c_1(\gamma)].$$

Considering the compensating variation Δ , welfare at zero inflation is denoted

$$\mathcal{W}_1 := u_2(\Delta c_2) - c_2 + \delta[u_1(\Delta c_1(1)) - c_1].$$

The welfare cost of $\gamma - 1$ inflation is the value $1 - \Delta$ where Δ satisfies $\mathcal{W}_1 - \mathcal{W}_\gamma = 0$.

The markup: In LW the markup varies with the bargaining power and it generally varies with c_1 (but not always; consider $\eta(h) = \frac{h^x}{x}$, $x \geq 1$ and $\theta = 1$). In the calibration labor disutility is linear so the markup coincides with the relative price $\frac{p_1}{p_2}$, which is $\frac{z(c_1; \theta)}{c_1}$.

The share of DM output : The share of DM output in LW is easily constructed, given that in the calibrated model everyone is matched in the DM ($\alpha = 1$ in LW).

DM output is δc_1 and CM output is $c_2 \equiv B$, in the calibrated model. Hence, total output is $Y = \delta c_1 + B$ and the DM output share is $\frac{\delta c_1}{Y}$ (it increases as inflation falls because real money balances increase); this also gives us the share of cash goods to total goods in the CIA model. This share is used to calculate average markups.

In the calibration, when $\theta = 0.5$ we have $\tau = \psi(c_1; \theta) = .719, .846, .928$ for, respectively, $\gamma = .1, 0, \frac{1-\beta}{\beta}$; the corresponding average sales tax rates are: .025, .037, .034. Instead, when $\theta = 0.343$, we have $\tau = \psi(c_1; \theta) = .511, .672, .802$; the corresponding average sales tax rates are: .014, .019, .013. As inflation decreases the markup in cash trades, $\frac{1}{\tau}$, falls; yet, the average markup increases because the share of cash goods to total output rises.