

Online Appendix

We begin by presenting two lemmas that are useful in our proof of Theorem 1. Fix any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$. Let $\mathbf{w}_{\alpha, E, E_s}$ denote the portfolio with minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s .

Lemma 1. For any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$, $\mathbf{w}_{\alpha, E, E_s}$ solves:

$$\min_{\mathbf{w} \in \mathbb{R}^N} \frac{1}{2} \mathbf{w}' \boldsymbol{\Sigma}_s \mathbf{w} \quad (8)$$

$$\mathbf{w}' \mathbf{1} = 1 \quad (9)$$

$$\mathbf{w}' \boldsymbol{\mu} = E \quad (10)$$

$$\mathbf{w}' \boldsymbol{\mu}_s = E_s. \quad (11)$$

Proof. Fix any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$. Suppose by way of a contradiction that $\mathbf{w}_{\alpha, E, E_s}$ does not solve minimization problem (8) subject to constraints (9)–(11). Then, there is a portfolio \mathbf{w}^* with:

$$\sigma_s^2[r_{\mathbf{w}^*}] < \sigma_s^2[r_{\mathbf{w}_{\alpha, E, E_s}}], \quad (12)$$

$$E[r_{\mathbf{w}^*}] = E, \quad (13)$$

$$E_s[r_{\mathbf{w}^*}] = E_s. \quad (14)$$

Using Eqs. (4), (12), and (14), we have:

$$C_{s, \alpha}[r_{\mathbf{w}^*}] < C_{s, \alpha}[r_{\mathbf{w}_{\alpha, E, E_s}}]. \quad (15)$$

Eqs. (13)–(15) contradict the fact that $\mathbf{w}_{\alpha, E, E_s}$ has minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s . ■

Fix any $(E, E_s) \in \mathbb{R}^2$. Let \mathbf{w}_{E, E_s} denote the portfolio that solves minimization problem (8) subject to constraints (9)–(11).

Lemma 2. For any $(E, E_s) \in \mathbb{R}^2$, we have:

$$\sigma_s^2[r_{\mathbf{w}_{E, E_s}}] = h_s + \frac{(E_s[r_{\mathbf{w}_{E, E_s}}] - i_s)^2}{g_s}, \quad (16)$$

where g_s is defined in Section 2, $h_s \equiv \frac{c_s E^2 - 2a_s E + b_s}{b_s c_s - a_s^2}$, and $i_s \equiv \frac{b_s d_s - a_s f_s + (c_s f_s - a_s d_s) E}{b_s c_s - a_s^2}$.⁴² Also, $g_s \in \mathbb{R}_{++}$ and $h_s \in \mathbb{R}_{++}$.

Proof. Fix any $(E, E_s) \in \mathbb{R}^2$. First, we show that Eq. (16) holds. A first-order condition for \mathbf{w}_{E, E_s} to solve minimization problem (8) subject to constraints (9)–(11) is:

$$\boldsymbol{\Sigma}_s \mathbf{w}_{E, E_s} - \lambda_1 \mathbf{1} - \lambda_2 \boldsymbol{\mu} - \lambda_3 \boldsymbol{\mu}_s = \mathbf{0}, \quad (17)$$

where λ_1 , λ_2 , and λ_3 are multipliers associated with such constraints. Since $\text{rank}(\boldsymbol{\Sigma}_s) = N$, Eq. (17) implies that:

$$\mathbf{w}_{E, E_s} = \lambda_1 \boldsymbol{\Sigma}_s^{-1} \mathbf{1} + \lambda_2 \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\mu} + \lambda_3 \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\mu}_s. \quad (18)$$

Premultiplying Eq. (18) by $\mathbf{1}'$ and using the definitions of a_s , c_s , and d_s as well as Eq. (9), we have:

$$\lambda_1 c_s + \lambda_2 a_s + \lambda_3 d_s = 1. \quad (19)$$

Similarly, premultiplying Eq. (18) by $\boldsymbol{\mu}'$ and using the definitions of a_s , b_s , and f_s as well as Eq. (10), we have:

$$\lambda_1 a_s + \lambda_2 b_s + \lambda_3 f_s = E. \quad (20)$$

Also, premultiplying Eq. (18) by $\boldsymbol{\mu}'_s$ and using the definitions of d_s , e_s , and f_s as well as Eq. (11), we have:

$$\lambda_1 d_s + \lambda_2 f_s + \lambda_3 e_s = E_s. \quad (21)$$

Using Eqs. (19)–(21) and elementary algebra, we obtain:

$$\lambda_1 = \frac{(b_s e_s - f_s^2) + (d_s f_s - a_s e_s) E + (a_s f_s - b_s d_s) E_s}{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + (b_s c_s - a_s^2) e_s}, \quad (22)$$

$$\lambda_2 = \frac{(d_s f_s - a_s e_s) + (c_s e_s - d_s^2) E + (a_s d_s - c_s f_s) E_s}{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + (b_s c_s - a_s^2) e_s}, \quad (23)$$

$$\lambda_3 = \frac{(a_s f_s - b_s d_s) + (a_s d_s - c_s f_s) E + (b_s c_s - a_s^2) E_s}{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + (b_s c_s - a_s^2) e_s}. \quad (24)$$

⁴² While h_s depends on E , we write ' h_s ' instead of ' $h_{s,E}$ ' for brevity. A similar remark applies to i_s .

It follows from Eq. (18) that:

$$\sigma_s^2[r_{\mathbf{w}_{E,E_s}}] = (\lambda_1 \Sigma_s^{-1} \mathbf{1} + \lambda_2 \Sigma_s^{-1} \boldsymbol{\mu} + \lambda_3 \Sigma_s^{-1} \boldsymbol{\mu}_s)' (\lambda_1 \mathbf{1} + \lambda_2 \boldsymbol{\mu} + \lambda_3 \boldsymbol{\mu}_s). \quad (25)$$

Using Eq. (25) as well as the definitions of a_s , b_s , c_s , d_s , e_s , and f_s , we obtain:

$$\sigma_s^2[r_{\mathbf{w}_{E,E_s}}] = \lambda_1^2 c_s + \lambda_2^2 b_s + \lambda_3^2 e_s + 2\lambda_1 \lambda_2 a_s + 2\lambda_1 \lambda_3 d_s + 2\lambda_2 \lambda_3 f_s. \quad (26)$$

Eqs. (22)–(24) and (26) along with elementary algebra imply that:

$$\sigma_s^2[r_{\mathbf{w}_{E,E_s}}] = \frac{c_s E^2 - 2a_s E + b_s}{b_s c_s - a_s^2} + \frac{\left[E_s - \frac{b_s d_s - a_s f_s + (c_s f_s - a_s d_s) E}{b_s c_s - a_s^2} \right]^2}{\frac{2a_s d_s f_s - b_s d_s^2 - c_s f_s^2}{b_s c_s - a_s^2} + e_s}. \quad (27)$$

Eq. (16) follows from the definitions of g_s , h_s , and i_s along with Eq. (27).

Second, we show that $g_s > 0$. Let $\Theta_2 \equiv \Psi_2' \Sigma_s^{-1} \Psi_2$ and $\Theta_3 \equiv \Psi_3' \Sigma_s^{-1} \Psi_3$ where $\Psi_2 \equiv [\mathbf{1} \quad \boldsymbol{\mu}]$ and $\Psi_3 \equiv [\mathbf{1} \quad \boldsymbol{\mu} \quad \boldsymbol{\mu}_s]$. Observe that $\det(\Theta_2) = b_s c_s - a_s^2$ and $\det(\Theta_3) = 2a_s d_s f_s - b_s d_s^2 - c_s f_s^2 + (b_s c_s - a_s^2) e_s$, where $\det(\cdot)$ denotes determinant. Noting that $g_s = \frac{\det(\Theta_3)}{\det(\Theta_2)}$, it suffices to show that $\det(\Theta_2) > 0$ and $\det(\Theta_3) > 0$. The fact that Σ_s is positive definite implies that Σ_s^{-1} is also positive definite. Since $\text{rank}(\Psi_2) = 2$ and Σ_s^{-1} is positive definite, Θ_2 is also positive definite. Hence, $\det(\Theta_2) > 0$. Similarly, since $\text{rank}(\Psi_3) = 3$ and Σ_s^{-1} is positive definite, Θ_3 is also positive definite. Hence, $\det(\Theta_3) > 0$.

Third, we show that $h_s > 0$. It suffices to show that the minimum value of h_s is positive. Note that $\frac{\partial h_s}{\partial E} = \frac{2(c_s E - a_s)}{b_s c_s - a_s^2}$. Since $\frac{\partial^2 h_s}{\partial E^2} = \frac{2c_s}{b_s c_s - a_s^2}$, $c_s > 0$, and $b_s c_s - a_s^2 > 0$, we have $\frac{\partial^2 h_s}{\partial E^2} > 0$. Hence, the minimum value of h_s occurs when $\frac{\partial h_s}{\partial E} = 0$ or $E = \frac{a_s}{c_s}$. It follows that the minimum value of h_s is $\frac{1}{c_s}$. Since $c_s > 0$, we have $\frac{1}{c_s} > 0$. ■

Next, we provide proofs of Theorem 1 and Corollary 1.

Proof of Theorem 1. Fix any expected return $E \in \mathbb{R}$. First, suppose that $\alpha \leq \alpha_s$. Note that:

$$0 < y_\alpha \leq \sqrt{g_s}. \quad (28)$$

In order to prove that no portfolio is on the M-SCVaR frontier, it suffices to show that $\frac{\partial C_{s,\alpha}[r_{w_{E,E_s}}]}{\partial E_s[r_{w_{E,E_s}}]} <$

0. Using Eqs. (4) and (16), we obtain:

$$\frac{\partial C_{s,\alpha}[r_{w_{E,E_s}}]}{\partial E_s[r_{w_{E,E_s}}]} = y_\alpha \frac{\frac{E_s[r_{w_{E,E_s}}] - i_s}{g_s}}{\sqrt{h_s + \frac{(E_s[r_{w_{E,E_s}}] - i_s)^2}{g_s}}} - 1. \quad (29)$$

Since $h_s > 0$, Eqs. (28) and (29) imply that $\frac{\partial C_{s,\alpha}[r_{w_{E,E_s}}]}{\partial E_s[r_{w_{E,E_s}}]} < 0$. This completes the first part of our proof.

Second, suppose that $\alpha > \alpha_s$. Note that:

$$y_\alpha > \sqrt{g_s}. \quad (30)$$

Using Lemmas 1 and 2 along with Eq. (29), a first-order condition for E_s^{**} to solve $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{w_{\alpha,E,E_s}}]$

is:

$$y_\alpha \frac{\frac{E_s^{**} - i_s}{g_s}}{\sqrt{h_s + \frac{(E_s^{**} - i_s)^2}{g_s}}} - 1 = 0. \quad (31)$$

It follows from Eq. (30) and elementary algebra that:

$$E_s^{**} = i_s + \sqrt{\frac{h_s}{y_\alpha^2 - g_s}} g_s. \quad (32)$$

Using Eq. (29) and elementary algebra, we have:

$$\frac{\partial^2 C_{s,\alpha}[r_{w_{E,E_s}}]}{\partial (E_s[r_{w_{E,E_s}}])^2} = \frac{\frac{y_\alpha h_s}{g_s}}{\left[h_s + \frac{(E_s[r_{w_{E,E_s}}] - i_s)^2}{g_s} \right]^{3/2}}. \quad (33)$$

Since $y_\alpha > 0$, $h_s > 0$, and $g_s > 0$, Eq. (33) implies that $\frac{\partial^2 C_{s,\alpha}[r_{w_{E,E_s}}]}{\partial (E_s[r_{w_{E,E_s}}])^2} > 0$. Hence, E_s^{**} solves $\min_{E_s \in \mathbb{R}}$

$C_{s,\alpha}[r_{w_{\alpha,E,E_s}}]$. Using Eqs. (22)–(24) with $E_s = E_s^{**}$ and elementary algebra, we have:

$$\lambda_1 = \frac{1}{b_s c_s - a_s^2} \left[(b_s - a_s E) + (a_s f_s - b_s d_s) \sqrt{\frac{h_s}{y_\alpha^2 - g_s}} \right], \quad (34)$$

$$\lambda_2 = \frac{1}{b_s c_s - a_s^2} \left[(c_s E - a_s) + (a_s d_s - c_s f_s) \sqrt{\frac{h_s}{y_\alpha^2 - g_s}} \right], \quad (35)$$

$$\lambda_3 = \sqrt{\frac{h_s}{y_\alpha^2 - g_s}}. \quad (36)$$

Let:

$$\theta_{0,\alpha,E} \equiv c_s \lambda_1, \quad (37)$$

$$\theta_{1,\alpha,E} \equiv a_s \lambda_2. \quad (38)$$

Using Eqs. (34)–(38), we obtain:

$$1 - \theta_{0,\alpha,E} - \theta_{1,\alpha,E} = d_s \lambda_3. \quad (39)$$

Eq. (5) follows from Eqs. (18) and (34)–(39). This completes the second part of our proof. ■

Proof of Corollary 1. Fix any expected return $E \in \mathbb{R}$. Since $h_s > 0$, we have:

$$\sqrt{\frac{h_s}{y_\alpha^2 - g_s}} \rightarrow 0 \text{ as } \alpha \rightarrow 1. \quad (40)$$

Let $\theta_{0,E} \equiv \left(\frac{b_s - a_s E}{b_s c_s - a_s^2} \right) c_s$ and $\theta_{1,E} \equiv \left(\frac{c_s E - a_s}{b_s c_s - a_s^2} \right) a_s$. Eqs. (34), (37), and (40) imply that $\theta_{0,\alpha,E} \rightarrow \theta_{0,E}$ as $\alpha \rightarrow 1$. Similarly, Eqs. (35), (38), and (40) imply that $\theta_{1,\alpha,E} \rightarrow \theta_{1,E}$ as $\alpha \rightarrow 1$. Also, Eqs. (36), (39), and (40) imply that $1 - \theta_{0,\alpha,E} - \theta_{1,\alpha,E} \rightarrow 0$ as $\alpha \rightarrow 1$. ■

We now present two lemmas that are useful in our proof of Theorem 2. Fix any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$. Let $\bar{\mathbf{w}}_{\alpha,E,E_s}$ denote the portfolio with minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s .

Lemma 3. For any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$, $\underline{\mathbf{w}}_{\alpha,E,E_s}$ solves:

$$\min_{\mathbf{w} \in \mathbb{R}^N} \frac{1}{2} \mathbf{w}' \boldsymbol{\Sigma}_s \mathbf{w} \quad (41)$$

$$\mathbf{w}' (\boldsymbol{\mu} - \mathbf{1} r_f) = E - r_f \quad (42)$$

$$\mathbf{w}' (\boldsymbol{\mu}_s - \mathbf{1} r_{f,s}) = E_s - r_{f,s}. \quad (43)$$

Proof. Fix any $(\alpha, E, E_s) \in (0.5, 1) \times \mathbb{R}^2$. Suppose by way of a contradiction that $\underline{\mathbf{w}}_{\alpha,E,E_s}$ does not solve minimization problem (41) subject to constraints (42) and (43). Then, there is a portfolio $\bar{\mathbf{w}}^*$ with:

$$\sigma_s^2[r_{\bar{\mathbf{w}}^*}] < \sigma_s^2[r_{\bar{\mathbf{w}}_{\alpha,E,E_s}}], \quad (44)$$

$$E[r_{\bar{\mathbf{w}}^*}] = E, \quad (45)$$

$$E_s[r_{\bar{\mathbf{w}}^*}] = E_s. \quad (46)$$

Using Eqs. (4), (44), and (46), we have:

$$C_{s,\alpha}[r_{\bar{\mathbf{w}}^*}] < C_{s,\alpha}[r_{\bar{\mathbf{w}}_{\alpha,E,E_s}}]. \quad (47)$$

Eqs. (45)–(47) contradict the fact that $\bar{\mathbf{w}}_{\alpha,E,E_s}$ has minimum SCVaR at the $100\alpha\%$ confidence level among all portfolios with an expected return of E and a stressed expected return of E_s . ■

Fix any $(E, E_s) \in \mathbb{R}^2$. Let $\bar{\mathbf{w}}_{E,E_s}$ denote the portfolio that solves minimization problem (41) subject to constraints (42) and (43).

Lemma 4. *For any $(E, E_s) \in \mathbb{R}^2$, we have:*

$$\sigma_s^2[r_{\bar{\mathbf{w}}_{E,E_s}}] = \bar{h}_s + \frac{(E_s[r_{\bar{\mathbf{w}}_{E,E_s}}] - \bar{i}_s)^2}{\bar{g}_s}, \quad (48)$$

where \bar{g}_s is defined in Section 3, $\bar{h}_s \equiv \frac{1}{j_s} (E - r_f)^2$, and $\bar{i}_s \equiv r_{f,s} + \frac{k_s}{j_s} (E - r_f)$.⁴³ Also, $\bar{g}_s \in \mathbb{R}_{++}$ and $\bar{h}_s \in \mathbb{R}_+$.

Proof. Fix any $(E, E_s) \in \mathbb{R}^2$. First, we show that Eq. (48) holds. A first-order condition for $\underline{\mathbf{w}}_{E,E_s}$ to solve minimization problem (41) subject to constraints (42) and (43) is:

$$\boldsymbol{\Sigma}_s \underline{\mathbf{w}}_{E,E_s} - \lambda_4 (\boldsymbol{\mu} - \mathbf{1}r_f) - \lambda_5 (\boldsymbol{\mu}_s - \mathbf{1}r_{f,s}) = \mathbf{0}, \quad (49)$$

where λ_4 and λ_5 are multipliers associated with such constraints. Since $\text{rank}(\boldsymbol{\Sigma}_s) = N$, Eq. (49) implies that:

$$\underline{\mathbf{w}}_{E,E_s} = \lambda_4 \boldsymbol{\Sigma}_s^{-1} (\boldsymbol{\mu} - \mathbf{1}r_f) + \lambda_5 \boldsymbol{\Sigma}_s^{-1} (\boldsymbol{\mu}_s - \mathbf{1}r_{f,s}). \quad (50)$$

Premultiplying Eq. (50) by $(\boldsymbol{\mu} - \mathbf{1}r_f)'$ and using the definitions of j_s and k_s as well as Eq. (42), we have:

$$\lambda_4 j_s + \lambda_5 k_s = E - r_f. \quad (51)$$

Similarly, premultiplying Eq. (50) by $(\boldsymbol{\mu}_s - \mathbf{1}r_{f,s})'$ and using the definitions of k_s and l_s as well as Eq. (43), we have:

$$\lambda_4 k_s + \lambda_5 l_s = E_s - r_{f,s}. \quad (52)$$

⁴³ While \bar{h}_s depends on E , we write ' \bar{h}_s ' instead of ' $\bar{h}_{s,E}$ ' for brevity. A similar remark applies to \bar{i}_s .

Using Eqs. (51) and (52) along with elementary algebra, we obtain:

$$\lambda_4 = \frac{l_s(E - r_f) - k_s(E_s - r_{f,s})}{j_s l_s - k_s^2}, \quad (53)$$

$$\lambda_5 = \frac{j_s(E_s - r_{f,s}) - k_s(E - r_f)}{j_s l_s - k_s^2}. \quad (54)$$

It follows from Eq. (50) that:

$$\sigma_s^2[r_{\bar{w}_{E,E_s}}] = [\lambda_4 \Sigma_s^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f) + \lambda_5 \Sigma_s^{-1}(\boldsymbol{\mu}_s - \mathbf{1}r_{f,s})]' [\lambda_4(\boldsymbol{\mu} - \mathbf{1}r_f) + \lambda_5(\boldsymbol{\mu}_s - \mathbf{1}r_{f,s})]. \quad (55)$$

Using Eq. (55) as well as the definitions of j_s , k_s , and l_s , we obtain:

$$\sigma_s^2[r_{\bar{w}_{E,E_s}}] = \lambda_4^2 j_s + \lambda_5^2 l_s + 2\lambda_4 \lambda_5 k_s. \quad (56)$$

Eqs. (53), (54), and (56) along with elementary algebra imply that:

$$\sigma_s^2[r_{\bar{w}_{E,E_s}}] = \frac{1}{j_s} (E - r_f)^2 + \frac{\left(E_s[r_{\mathbf{w}_{E,E_s}}] - \left[r_{f,s} + \frac{k_s}{j_s}(E - r_f)\right]\right)^2}{l_s - k_s^2/j_s}. \quad (57)$$

Eq. (48) follows from the definitions of \bar{g}_s , \bar{h}_s , and \bar{i}_s along with Eq. (57).

Second, we show that $\bar{g}_s > 0$. Let $\bar{\Theta}_2 \equiv \bar{\Psi}_2' \Sigma_s^{-1} \bar{\Psi}_2$ where $\bar{\Psi}_2 \equiv [\boldsymbol{\mu}_s - \mathbf{1}r_{f,s} \quad \boldsymbol{\mu} - \mathbf{1}r_f]$. Observe that $\det(\bar{\Theta}_2) = l_s j_s - k_s^2$. Noting that $\bar{g}_s = \frac{\det(\bar{\Theta}_2)}{j_s}$, it suffices to show that $\det(\bar{\Theta}_2) > 0$ and $j_s > 0$. Since $\text{rank}([\mathbf{1} \quad \boldsymbol{\mu} \quad \boldsymbol{\mu}_s]) = 3$, we have $\text{rank}(\bar{\Psi}_2) = 2$. Since $\text{rank}(\bar{\Psi}_2) = 2$ and Σ_s^{-1} is positive definite, $\bar{\Theta}_2$ is also positive definite. Hence, $\det(\bar{\Theta}_2) > 0$. Observe that $j_s = (\boldsymbol{\mu} - \mathbf{1}r_f)' \Sigma_s^{-1} (\boldsymbol{\mu} - \mathbf{1}r_f)$. Noting that $\text{rank}([\mathbf{1} \quad \boldsymbol{\mu}]) = 2$, we have $\boldsymbol{\mu} - \mathbf{1}r_f \neq \mathbf{0}$. Since Σ_s^{-1} is positive definite and $\boldsymbol{\mu} - \mathbf{1}r_f \neq \mathbf{0}$, we have $j_s > 0$.

Third, we show that $\bar{h}_s \geq 0$. Since $\bar{h}_s = \frac{1}{j_s} (E - r_f)^2$ and $j_s > 0$, we have $\bar{h}_s \geq 0$. ■

Next, we provide proofs of Theorem 2 and Corollary 2.

Proof of Theorem 2. Fix any expected return $E \in \mathbb{R}$. First, suppose that $\alpha < \bar{\alpha}_s$. Note that:

$$0 < y_\alpha < \sqrt{\bar{g}_s}. \quad (58)$$

In order to prove that no portfolio is on the M-SCVaR frontier at the 100 α % confidence level, it suffices to show that $\frac{\partial C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial E_s[r_{\bar{w}_{E,E_s}}]} < 0$. Using Eqs. (4) and (48), we obtain:

$$\frac{\partial C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial E_s[r_{\bar{w}_{E,E_s}}]} = y_\alpha \frac{\frac{E_s[r_{\bar{w}_{E,E_s}}] - \bar{i}_s}{\bar{g}_s}}{\sqrt{\bar{h}_s + \frac{(E_s[r_{\bar{w}_{E,E_s}}] - \bar{i}_s)^2}{\bar{g}_s}}} - 1. \quad (59)$$

Since $\bar{h}_s \geq 0$, Eqs. (58) and (59) imply that $\frac{\partial C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial E_s[r_{\bar{w}_{E,E_s}}]} < 0$. This completes the first part of our proof.

Second, suppose that $\alpha = \bar{\alpha}_s$ and $E \neq r_f$. Note that:

$$y_\alpha = \sqrt{\bar{g}_s}. \quad (60)$$

and:

$$\bar{h}_s > 0. \quad (61)$$

In order to prove that no portfolio is on the M-SCVaR frontier at the 100 α % confidence level, it suffices to show that $\frac{\partial C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial E_s[r_{\bar{w}_{E,E_s}}]} < 0$. Eqs. (59)–(61) imply that $\frac{\partial C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial E_s[r_{\bar{w}_{E,E_s}}]} < 0$. This completes the second part of our proof.

Third, suppose that $\alpha = \bar{\alpha}_s$ and $E = r_f$. Note that Eq. (60) holds and:

$$\bar{h}_s = 0. \quad (62)$$

Using Eqs. (48), (60), and (62), we have:

$$\sigma_s[r_{\bar{w}_{E,E_s}}] = \begin{cases} \frac{E_s[r_{\bar{w}_{E,E_s}}] - r_{f,s}}{\sqrt{\bar{g}_s}} & \Leftarrow E_s[r_{\bar{w}_{E,E_s}}] > r_{f,s} \\ -\frac{E_s[r_{\bar{w}_{E,E_s}}] - r_{f,s}}{\sqrt{\bar{g}_s}} & \Leftarrow E_s[r_{\bar{w}_{E,E_s}}] < r_{f,s} \end{cases}. \quad (63)$$

It follows from Eqs. (4), (60), and (63) that:

$$\frac{\partial C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial E_s[r_{\bar{w}_{E,E_s}}]} = \begin{cases} 0 & \Leftarrow E_s[r_{\bar{w}_{E,E_s}}] > r_{f,s} \\ -2 & \Leftarrow E_s[r_{\bar{w}_{E,E_s}}] < r_{f,s} \end{cases}. \quad (64)$$

Lemmas 3 and 4 along with Eq. (64) imply that for any $E_s \geq r_{f,s}$ portfolio \bar{w}_{E,E_s} is on the M-SCVaR frontier at the 100 α % confidence level. Note that $\bar{w}_{E,E_s} = \bar{w}_0$ if $E = r_f$ and $E_s = r_{f,s}$.

This completes the third part of our proof.

Fourth, suppose that $\alpha > \bar{\alpha}_s$. Note that:

$$y_\alpha > \sqrt{\bar{g}_s}. \quad (65)$$

Assume that $E \neq r_f$. Then, Eq. (61) holds. Using Lemmas 3 and 4 along with Eq. (59), a first-order condition for \bar{E}_s^{**} to solve $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]$ is:

$$y_\alpha \frac{\frac{\bar{E}_s^{**} - \bar{i}_s}{\bar{g}_s}}{\sqrt{\bar{h}_s + \frac{(\bar{E}_s^{**} - \bar{i}_s)^2}{\bar{g}_s}}} - 1 = 0. \quad (66)$$

It follows from Eq. (66) and elementary algebra that:

$$E_s^{**} = \bar{i}_s + \sqrt{\frac{\bar{h}_s}{y_\alpha^2 - \bar{g}_s} \bar{g}_s}. \quad (67)$$

Using Eq. (59) and elementary algebra, we have:

$$\frac{\partial^2 C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial (E_s[r_{\bar{w}_{E,E_s}}])^2} = \frac{\frac{y_\alpha \bar{h}_s}{\bar{g}_s}}{\left[\bar{h}_s + \frac{(E_s[r_{\bar{w}_{E,E_s}}] - \bar{i}_s)^2}{\bar{g}_s} \right]^{3/2}}. \quad (68)$$

Since $y_\alpha > 0$, $\bar{g}_s > 0$, and $\bar{h}_s > 0$, Eq. (68) implies that $\frac{\partial^2 C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]}{\partial (E_s[r_{\bar{w}_{E,E_s}}])^2} > 0$. Hence, \bar{E}_s^{**} solves $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r_{\bar{w}_{E,E_s}}]$. Using Eqs. (53) and (54) with $E_s = \bar{E}_s^{**}$, the definitions of \bar{g}_s , \bar{h}_s , and \bar{i}_s , and elementary algebra, we have:

$$\lambda_4 = \frac{1}{j_s} \left[(E - r_f) - k_s \sqrt{\frac{\frac{1}{j_s} (E - r_f)^2}{y_\alpha^2 - \bar{g}_s}} \right], \quad (69)$$

$$\lambda_5 = \sqrt{\frac{\frac{1}{j_s} (E - r_f)^2}{y_\alpha^2 - \bar{g}_s}}. \quad (70)$$

Let:

$$\bar{\theta}_{0,\alpha,E} \equiv 1 - (a_s - c_s r_f) \lambda_4 - (d_s - c_s r_{f,s}) \lambda_5, \quad (71)$$

$$\bar{\theta}_{1,\alpha,E} \equiv (a_s - c_s r_f) \lambda_4. \quad (72)$$

Using Eqs. (69)–(72), we obtain:

$$1 - \bar{\theta}_{0,\alpha,E} - \bar{\theta}_{1,\alpha,E} = (d_s - c_s r_{f,s}) \lambda_5. \quad (73)$$

Eq. (7) follows from Eqs. (50) and (69)–(73).⁴⁴

Assume that $E = r_f$. Then, Eq. (63) holds. Using Eqs. (63) and (65), we have $\frac{\partial C_{s,\alpha}[r\bar{w}_{E,E_s}]}{\partial E_s[r\bar{w}_{E,E_s}]} < 0$ if $E_s < r_{f,s}$ and $\frac{\partial C_{s,\alpha}[r\bar{w}_{E,E_s}]}{\partial E_s[r\bar{w}_{E,E_s}]} > 0$ if $E_s > r_{f,s}$. Hence, $\bar{E}_s^{**} = r_{f,s}$ solves $\min_{E_s \in \mathbb{R}} C_{s,\alpha}[r\bar{w}_{E,E_s}]$. It follows that Eq. (7) holds with $\bar{\theta}_{0,\alpha,E} = 1$ and $\bar{\theta}_{1,\alpha,E} = 0$. This completes the fourth part of our proof. ■

Proof of Corollary 2. Fix any expected return $E \in \mathbb{R}$. Let $\bar{\theta}_{0,E} \equiv 1 - \frac{1}{j_s} (a_s - c_s r_f) (E - r_f)$ and $\bar{\theta}_{1,E} \equiv \frac{1}{j_s} (a_s - c_s r_f) (E - r_f)$. First, suppose that $E = r_f$. It follows from the proof of Theorem 2 that $\bar{\theta}_{0,\alpha,E} = 1$, $\bar{\theta}_{1,\alpha,E} = 0$, and $1 - \bar{\theta}_{0,\alpha,E} - \bar{\theta}_{1,\alpha,E} = 0$. The desired claims follow from the fact that $\bar{\theta}_{0,E} = 1$ and $\bar{\theta}_{1,E} \equiv 0$.

Second, suppose that $E \neq r_f$. Note that:

$$\sqrt{\frac{\frac{1}{j_s} (E - r_f)^2}{y_\alpha^2 - \bar{g}_s}} \rightarrow 0 \text{ as } \alpha \rightarrow 1. \quad (74)$$

Eqs. (69), (70), and (74) imply that $\bar{\theta}_{0,\alpha,E} \rightarrow \bar{\theta}_{0,E}$ as $\alpha \rightarrow 1$. Similarly, Eqs. (69), (72), and (74) imply that $\bar{\theta}_{1,\alpha,E} \rightarrow \bar{\theta}_{1,E}$ as $\alpha \rightarrow 1$. Also, Eqs. (70), (73), and (74) imply that $1 - \bar{\theta}_{0,\alpha,E} - \bar{\theta}_{1,\alpha,E} \rightarrow 0$ as $\alpha \rightarrow 1$. ■

⁴⁴If $\alpha > \bar{\alpha}_s$, an alternative characterization of the composition of portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level is as follows. Suppose that $E < r_f$. Then, $\bar{w}_{\alpha,E} = \bar{\theta}_{0,\alpha,E} \bar{w}_0 + (1 - \bar{\theta}_{0,\alpha,E}) \bar{w}_{3,s}$ where $\bar{w}_{3,s} \equiv \psi \bar{w}_{1,s} + (1 - \psi) \bar{w}_{2,s}$, $\psi \equiv \frac{(1+k_s m_s)(a_s - c_s r_f)/j_s}{(1+k_s m_s)(a_s - c_s r_f)/j_s - m_s(d_s - c_s r_{f,s})}$, and $m_s \equiv \sqrt{\frac{1/j_s}{y_\alpha^2 - \bar{g}_s}}$. Hence, portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level with expected returns smaller than r_f exhibit two-fund separation with the two funds being \bar{w}_0 and $\bar{w}_{3,s}$. Similarly, suppose that $E \geq r_f$. Then, $\bar{w}_{\alpha,E} = \bar{\theta}_{0,\alpha,E} \bar{w}_0 + (1 - \bar{\theta}_{0,\alpha,E}) \bar{w}_{4,s}$ where $\bar{w}_{4,s} \equiv \varphi \bar{w}_{1,s} + (1 - \varphi) \bar{w}_{2,s}$, and $\varphi \equiv \frac{(1-k_s m_s)(a_s - c_s r_f)/j_s}{(1-k_s m_s)(a_s - c_s r_f)/j_s + m_s(d_s - c_s r_{f,s})}$. Hence, portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level with expected returns equal to or larger than r_f exhibit two-fund separation with the two funds being \bar{w}_0 and $\bar{w}_{4,s}$. Since $\bar{w}_{3,s}$ generally differs from $\bar{w}_{4,s}$, portfolios on the M-SCVaR frontier at the $100\alpha\%$ confidence level with all expected returns exhibit three-fund separation with the three funds being \bar{w}_0 , $\bar{w}_{3,s}$, and $\bar{w}_{4,s}$ (or, equivalently, \bar{w}_0 , $\bar{w}_{1,s}$, and $\bar{w}_{2,s}$).