

ON-LINE APPENDIX FOR

“Interventions in markets with adverse selection: Implications for discount window stigma”

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1. PROOFS

Lemma 1. In any equilibrium with an active market for private loans, we have that $r^*(l, m) > 1$ for all $l > 0$ and all m .

Proof. The result follows from applying the break-even condition and the fact that $\rho(\theta, rl) < rl$ whenever $rl > 0$ since we then have that:

$$l = \int_{\Theta} \rho[\theta, r^*(l, m)l] dB^*(\theta | l, m) < \int_{\Theta} r^*(l, m)l dB^*(\theta | l, m) = r^*(l, m)l,$$

which implies that $r^*(l, m) > 1$. ■

Lemma 2. For any $\theta \in \Theta$, if $i^*(\theta) = 0$ then $l^*(\theta) = 0$ and, when $R > 1$, also $m^*(\theta) = 0$.

Proof. From Lemma 1 we know that $r^*(l; m) \equiv r_{lm}^* > 1$ for all $l > 0$ and all m . Now suppose, by way of contradiction, that for some θ we have that $i^*(\theta) = 0$ while $l^*(\theta) > 0$ and/or $m^*(\theta) > 0$. In that case, the payoff of the firm is:

$$\int_{\underline{a}}^{\bar{A}} (c_0 + l + m + a - r_{lm}^*l - Rm) f_A(a | \theta) da,$$

where $\underline{a} = \max\{0, r_{lm}^*l + Rm - c_0 + l + m\}$. But, then, we have that:

$$\int_{\underline{a}}^{\bar{A}} [c_0 + a - (r_{lm}^* - 1)l - (R - 1)m] f_A(a | \theta) da < \int_A (c_0 + a) f_A(a | \theta) da,$$

where the last term is the payoff of the firm that does not borrow and does not invest. Hence, $l^*(\theta) = 0$ and $m^*(\theta) = 0$ when $i^*(\theta) = 0$ is a better option for firms. Since this strategy is always available to firms, the statement of the lemma must hold. ■

Proposition 1. Equilibrium without discount window. When the discount window is not active, there is an equilibrium where: **(1)** $l^*(\theta) = l_0$ for all $\theta \leq \theta^* < \bar{\theta}$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^* < \bar{\theta}$ and zero otherwise; **(3)** the market interest rate is equal to r^* ; and **(4)** the market beliefs $B(\theta | l_0) = H(\theta)/H(\theta^*)$ for all $\theta \leq \theta^*$ and zero otherwise.

Proof. First, we show that θ^* lies in the interior of the set Θ . Suppose this is not the case and instead $\theta^* = \underline{\theta}$, then by equation (8) we have that $r^* = r_0$ and hence $\rho(\underline{\theta}, r^*l_0) = l_0$. But, then, since $E[v] > x$, this contradicts equation (7). Now suppose that $\theta^* = \bar{\theta}$, then equation (8) implies $r^* = \bar{r}_0$ and condition (3) (in Section 2.1) immediately implies a contradiction of equation (7). Clearly, for the pair $(\underline{\theta}, r_0)$ we have that $l_0 - x + E[v] - \rho(\underline{\theta}, r_0l_0) > 0$ and for the pair $(\bar{\theta}, \bar{r}_0)$ we have that $l_0 - x + E[v] - \rho(\bar{\theta}, \bar{r}_0l_0) < 0$. Since both expressions (7) and (8) are continuous in (θ^*, r^*) , we have that there is a solution to the system of equations (7) and (8) with $\theta^* \in (\underline{\theta}, \bar{\theta})$ and $r^* \in (\bar{r}_0, r_0)$.

To see that $l^*(\theta)$ and $i^*(\theta)$ are individually rational, first note that by Lemma 2 no firm would borrow to not invest. In consequence, the alternative to borrowing and investing is to not borrow and not invest. Now, note that to be able to invest, a firm needs to borrow from the market at least l_0 . If the firm borrows exactly l_0 when it decides to invest, then it would decide to invest whenever the following inequality holds:

$$\int_A \int_V [c_0 + l_0 - x + a + v - \min(a + v, r^*l_0)] f_V(v) f_A(a | \theta) dv da \geq \int_A (c_0 + a) f_A(a | \theta) da,$$

which can be simplified to:

$$l_0 - x + E[v] - \rho(\theta, r^*l_0) \geq 0. \quad (17)$$

Recall that ρ is a strictly increasing function of θ . Then, by the definition of θ^* in equation (7) we have that equation (17) holds for all $\theta \leq \theta^*$ and does not hold for any $\theta > \theta^*$. This confirms that conditional on a firm borrowing l_0 , the decision function $i^*(\theta)$ in the statement of the proposition satisfies individual rationality.

In principle, there are several specifications of off-equilibrium beliefs that can sustain $l^*(\theta)$ as an equilibrium. A simple case is when beliefs are such that for all $l > l_0$ we have that $B(\theta | l) = 1$ if $\theta = \underline{\theta}$ and zero otherwise. That is, if a firm were to ask for a loan greater than l_0 , investors would believe that the firm is of type $\underline{\theta}$.²² Given these beliefs, the break-even condition implies that investors will charge an interest rate r_l that satisfies:

$$\int_Y \min(y + l - l_0, r_l l) f_Y(y | \underline{\theta}) dy = l.$$

Using that $\rho(\underline{\theta}, r_0l_0) = l_0$, it is easy to show that $r_0l_0 = r_l l_0 + (r_l - 1)(l - l_0)$ and the payoff to a firm of taking a loan $l > l_0$ at rate r_l is the same as the payoff to a firm of taking a loan l_0 at rate r_0 . Now, from expressions (7) and (8) we have that $\rho(\underline{\theta}, r^*l_0) < l_0$, which implies that $r^* < r_0$. Hence, taking a loan l_0 at rate r^* gives a higher payoff to the firm than taking a loan $l > l_0$ at rate r_l . We conclude that the decision function $l^*(\theta)$ in the statement of the proposition is individually rational under the proposed beliefs system. ■

²²Note that no firm would ask for a loan lower than l_0 because then investors would know that the firm is not investing and would demand a high interest rate, making borrowing not optimal for the firm.

Off-equilibrium beliefs are rather extreme in the proof of this proposition. In particular, investors believe that any firm asking for a loan greater than l_0 has legacy assets of the lowest type. This was used just for simplicity. The argument still goes through for many other systems of off-equilibrium beliefs. In fact, even if investors believe that any firm borrowing $l > l_0$ is a random draw from the relevant set of firms, the equilibrium configuration in Proposition 1 is still an equilibrium.

Formally, suppose that for any $l > l_0$ investors' beliefs are $B(\theta | l) = H(\theta)/H(\theta^*)$ if $\theta < \theta^*$ and zero otherwise. That is, investors believe that a firm borrowing $l > l_0$ is a random draw from the set of firms willing to borrow l and invest. Then, the break-even condition for investors is:

$$\int_{\theta}^{\theta^*} \int_Y \min(y + l - l_0, r_l l) f_Y(y | \theta) dy \frac{dH(\theta)}{H(\theta^*)} = l.$$

Subtracting $l - l_0$ from both sides of the equation, we get:

$$\int_{\theta}^{\theta^*} \int_Y \min(y, r_l l - (l - l_0)) f_Y(y | \theta) dy \frac{dH(\theta)}{H(\theta^*)} = l_0,$$

which can be compared with the break-even condition for loans of size l_0 :

$$\int_{\theta}^{\theta^*} \int_Y \min(y, r^* l_0) f_Y(y | \theta) dy \frac{dH(\theta)}{H(\theta^*)} = l_0,$$

to conclude that $r_l l - (l - l_0) = r^* l_0$.

The payoff of a firm borrowing $l > l_0$ at rate r_l is given by:

$$\int_Y (y + l - l_0 - \min\{y + l - l_0, r_l l\}) f_Y(y | \theta) dy = \int_Y (y - \min\{y, r_l l - (l - l_0)\}) f_Y(y | \theta) dy,$$

and the payoff of a firm borrowing l_0 at rate r^* is given by:

$$\int_Y [y - \min(y, r^* l_0)] f_Y(y | \theta) dy$$

but, since $r_l l - (l - l_0) = r^* l_0$, firms are indifferent between the two alternatives and choosing to borrow l_0 is an equilibrium.²³

Any belief system that is more “pessimistic” about the types of those firms borrowing more than l_0 , such as the one used in the proof of the proposition, will make firms no longer indifferent between borrowing l_0 or more. When this is the case, the equilibrium action in Proposition 1 becomes the strictly-preferred choice for firms.

Proposition 2. Philippon-Skreta equilibrium with a discount window. When the discount window offers loans of size l_0 at interest rate $R^T \in (1, r^D)$, there is an equilibrium where: **(1)** $m^*(\theta) = l_0$ for all $\theta < \theta^P$ and zero otherwise and $l^*(\theta) = l_0$ for all $\theta \in [\theta^P, \theta^T]$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise; **(3)** the market interest rate equals R^T ; and **(4)** the market beliefs $B(\theta | l_0, 0) = H(\theta)/[H(\theta^T) - H(\theta^P)]$ for all $\theta \in [\theta^P, \theta^T]$ and zero otherwise.

²³Note that considering an alternative equilibrium, where some randomly selected firms borrow l_0 and some borrow $l > l_0$, does not change equilibrium predictions in any significant way: it is still the case that firms with $\theta \leq \theta^*$ borrow and invest, and investors break even.

Proof. If the firm decides to invest, then it must pick l and m such that $l + m \geq l_0$. Consider the case when the firm investing chooses $l + m = l_0$. Given the equilibrium interest rate in the market, the payoff of a type- θ firm is:

$$\int [y - \min(y, R^T m + R^T l)] f_Y(y | \theta) dy,$$

and, hence, the payoffs from choosing $m = l_0$ or $l = l_0$ (with $l + m = l_0$) are the same.

Assume, as we did in the proof of Proposition 1, that off-equilibrium beliefs for $l > l_0$ and $m = 0$ are given by $B(\underline{\theta} | l, 0) = 1$. Then, just as in the proof of Proposition 1, break-even conditions imply that $r_l l - (l - l_0) = r_0 l_0$ and, since $R^T < r_0$, firms have no incentives to deviate and borrow more than l_0 when borrowing from the private market. When a discount window is available, we need to also consider the situation when $m = l_0$ and $l \neq l_0$. Again, assume that $B(\underline{\theta} | l, l_0) = 1$ in this case. Since the break-even condition implies that $r_{lm} > 1$, no firm will choose to deviate to $m = l_0$ and $l \neq l_0$.

Using Lemma 2, we have that a firm of type θ would choose $i^*(\theta) = 1$ if and only if:

$$\int [y - \min(y, R^T l_0)] f_Y(y | \theta) dy \geq \int (c_0 + a) f_A(a | \theta) dy,$$

which is equivalent to:

$$E[v] - \rho(\theta, R^T l_0) \geq c_0 = x - l_0.$$

Hence, from the definition of θ^T and the fact that $\rho(\theta, R^T l)$ is increasing in θ , we have that $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise.

Given that all firms with $\theta \leq \theta^T$ choose to invest and that firms that invest are indifferent between any feasible choice of l and m such that $l + m = l_0$, we have that $m^*(\theta) = l_0$ for $\theta \leq \theta^P$ and $l^*(\theta) = l_0$ for $\theta \in [\theta^P, \theta^T]$ satisfy individual rationality. Since only firms with $\theta \in [\theta^P, \theta^T]$ borrow from the market, belief consistency implies that $B(\theta | l_0, 0) = H(\theta)/[H(\theta^T) - H(\theta^P)]$ for all $\theta \in [\theta^P, \theta^T]$, as required. Finally, by the definition of θ^P in equation (10), the break-even condition is immediately satisfied. \blacksquare

Proposition 4. Equilibrium with flexible discount window lending. When the discount window offers loans of any size $m \leq l_0$ at an interest rate R , there is an equilibrium where: **(1)** $m^*(\theta) = m^{**}$ and $l^*(\theta) = l_0 - m^{**}$ for all $\theta \leq \theta^{**}$ and zero otherwise; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^{**}$ and zero otherwise; **(3)** the market interest-rate function is $r^*(l_0 - m, m) = r^{**}(m)$; **(4)** the market beliefs are: $B(\theta | l_0 - m, m) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ and all $m \leq l_0$.

Proof. Consider the following system of beliefs: for all $m \geq 0$ and $l = l_0 - m$, let $B(\theta | l, m) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ and zero otherwise; for all $m \geq 0$ and $l > l_0 - m$, let $B(\underline{\theta} | l, m) = 1$. Given these beliefs, for all $m \geq 0$ and $l = l_0 - m$, the break-even condition for investors (12) implies

that $r^*(l_0 - m, m) = r^{**}(m)$. If $l \geq l_0 - m$, then the break-even condition for investors under the proposed beliefs is:

$$\int_Y \min(y + l + m - l_0, r_{lm}l) f_Y(y | \theta) dy = l,$$

which determines the value of r_{lm} . Following similar steps as in the proof of the previous propositions, we can show that $r^{**}(m)(l_0 - m) < r_{lm}l - (l + m - l_0)$, which implies that the funding cost associated with a loan of size $l > l_0 - m$ is higher than the funding cost of a loan of size $l = l_0 - m$. Hence, firms take loans in the private market of size $l_0 - m$. From this we conclude that, given the pricing function $r^{**}(m)$ and the fact that m^{**} solves equation (13), firms investing will choose $m^*(\theta) = m^{**}$ and $l^*(\theta) = l_0 - m^{**}$. Using Lemma 2 and equation (14), we have that all firms with $\theta \leq \theta^{**}$, and only those firms, will choose to invest (that is, $i^*(\theta) = 1$ for all $\theta \leq \theta^{**}$ and zero otherwise). Finally, given firms decisions, we have that the beliefs $B(\theta | l_0 - m^{**}, m^{**}) = H(\theta)/H(\theta^{**})$ for all $\theta \leq \theta^{**}$ satisfy Bayes rule. ■

2. INACTIVE-MARKET EQUILIBRIUM

Suppose again that the interest rate at the discount window is $R^T \in (1, r^D)$ and the central bank offers loans of size l_0 . This discount window policy is *the same* as the one in place in the equilibrium described in Proposition 2. Interestingly, there is another equilibrium consistent with that policy, where all firms that borrow and invest get their funding from the discount window and the market for private loans is inactive.

Proposition. Equilibrium with an inactive private market. When the discount window offers loans of size l_0 at interest rate $R^T \in (1, r^D)$, there is an equilibrium where: **(1)** $m^*(\theta) = l_0$ for all $\theta \leq \theta^T$ and zero otherwise and $l^*(\theta) = 0$ for all $\theta \in \Theta$; **(2)** $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise; **(3)** the private market for loans is inactive.

Proof. Suppose firms borrowing in equilibrium only borrow from the discount window. We verify this is the case later in the proof. Since $R^T > 1$, firms only borrow from the discount window if they plan to invest. A firm planning to invest, then, borrows l_0 from the discount window at rate R^T . A firm would invest if and only if:

$$E[v] - \rho(\theta, R^T l_0) \geq c_0 = x - l_0.$$

Hence, given the definition of θ^T in equation (9) and the fact that $\rho(\theta, R^T l)$ is increasing in θ , we have that $i^*(\theta) = 1$ for all $\theta \leq \theta^T$ and zero otherwise.

It remains to verify that no firm would want to borrow from the private market. Assume that off-equilibrium beliefs are such that $B(\theta | l, m) = 1$ for all $l > 0$ and all m . There are two cases to consider. First, if a firm borrows l_0 from the discount window and some extra funds l from the market, then the firm will be able to pay back the private loan with probability one and the break-even interest rate is equal to one. The firm, then, is indifferent between playing the equilibrium strategy or deviating to this alternative. The second case is when the firm does not borrow from the discount window and instead takes a loan of size $l \geq l_0$ from the market. Following similar steps as in the proof of Proposition 1 we can show that the firm would be indifferent between taking a loan of size $l > l_0$ at rate \underline{r}_l and a loan of size l_0 at rate \underline{r}_0 . Since $\underline{r}_0 > R^T$, we have that the firm would prefer to take a loan of size l_0 at rate R^T from the discount window. ■

As before, there are other off-equilibrium beliefs consistent with this same equilibrium and, hence, it is not essential to have investors believing that any firm asking for a loan in the market has legacy assets of the lowest type. Even if investors believe that a firm borrowing from the market is a random draw from the relevant set of firms, the equilibrium configuration in the proposition is still an equilibrium. In this case, the “relevant set of firms” is those that would find the strategy of borrowing from the market and investing more attractive than not borrowing and not investing.

To understand this claim note that the break-even condition implies that the net interest cost for the firm of borrowing and investing is the same regardless of whether the firm borrows from the

market $l > l_0$ or exactly l_0 . From equations (7) and (8) in the paper, the relevant firms are those for which $\theta \leq \theta^D$ and the borrowing cost is $r^*l_0 = r^D l_0$. Given that $r^D > R^T$, firms will prefer to borrow from the discount window rather than from the market, which confirms the equilibrium of the proposition, where the private market is inactive.

Note, finally, that this equilibrium cannot be refined away using the intuitive criterion: if a firm deviates and borrows from the private market claiming to be a high- θ type and investors believe it, hence lowering the interest rate, then all other firms with lower values of θ would have similar incentives to deviate. This logic undermines the power of the intuitive criterion more generally in this model.

3. LIMITED MARKET ACCESS

Consider an extension of the model where each firm can access the market for private funds at time 1 only with some probability $\sigma < 1$. With probability $1 - \sigma$ the firm can only obtain funding by borrowing from the discount window —if it chooses to do so. At time 0 each firm finds out if it will have access to the market at time 1.

Given a discount window policy (m, R) , if $m < l_0$ then firms with no access to the market will not borrow at the discount window and the discount window could not serve as the backup source of funding that we are interested in considering here. So, assume that $m = l_0$.

If the discount rate is lower than the (expected) market interest rate then no firm borrows in the market and it makes no difference for firms to have or not have access to that market. On the other hand, if the discount rate is higher than the market interest rate, then, conditional on having access to the market, all firms with $\theta \leq \theta^*$ borrow from the market and θ^* solves equation (7) in the paper.

To determine the market interest rate, note that:

$$P(\theta \mid \text{borrowing}) = \begin{cases} \frac{\sigma P(\theta)}{P(\text{borrowing})} & \text{for } \theta \leq \theta^* \\ 0 & \text{otherwise,} \end{cases}$$

where $P(\text{borrowing}) = \sigma P(\theta \leq \theta^*)$. This implies that the conditional probability over types participating in the market is independent of the probability of having access to the market (σ). Then, we have that r^* solves equation (8) in the paper.

Firms with no access to the market borrow from the discount window as long as $\theta \leq \theta^R$ where θ^R solves:

$$l_0 - x + E[v] - \rho(\theta^R, Rl_0) = 0,$$

which is equivalent to equation (9) in the paper.

If the discount rate is equal to the market interest rate, then firms that have access to the market are indifferent between borrowing in the market or at the discount window and we can construct an equilibrium like the one in Proposition (2). In particular, suppose that $R = R^T$ and consider θ^T and θ^P , the solutions to equations (9) and (10). Here, again, the conditional probabilities are independent of the probability of having access to the market. Then, all firms with $\theta \leq \theta^P$ borrow from the discount window regardless of their access to private liquidity and those firms with $\theta \in [\theta^P, \theta^T]$ borrow from the private market if they have access to it, and from the discount window if they do not.