Inequalities and Business Cycles
in Dynamic Stochastic General Equilibrium Models*

Jonghyeon Oh†

December 3, 2013

Job Market Paper

Abstract
Consumption dispersion and dynamics of inequality are the important aspects of economies but largely overlooked in the study of business cycles. Examining the Gini coefficients for earnings, income, and consumption, I observe two stylized facts: (1) consumption inequality is significant, and (2) inequality is moderately counter-cyclical at an annual frequency. To explain the facts, a dynamic stochastic general equilibrium model with heterogeneous households is proposed. The model has three main ingredients: (1) non-separable preferences between consumption and leisure eliminating wealth effects on labor supply, (2) time-varying dispersion of idiosyncratic labor productivity, and (3) time-varying threshold of idiosyncratic labor productivity for employment. The preference specification is necessary to explain the significant disparity in consumption. The last two elements are required for inequality to be moderately counter-cyclical.

Keywords: Time-varying dispersion, Earnings inequality, Income inequality, Consumption inequality, Idiosyncratic labor income risks

JEL Classification: C68, D39, E32

*I would like to thank Julia Thomas and Aubhik Khan for their helpful comments, support and guidance. I am also grateful to Paul Evans and other seminar participants at the Ohio State University. This work was supported in part by an allocation of computing time from the Ohio Supercomputer Center. All remaining errors are mine.

†Department of Economics, The Ohio State University, Columbus, OH 43210, oh.183@osu.edu
1 Introduction

Consumption inequality is one of the most important measures of disparity in welfare because consumption itself, rather than wealth or income, directly determines the utility of individuals. In spite of its importance, the literature on heterogeneous agent models does not pay much attention to consumption inequality. An important class of the models is heterogeneous agent economies with partially insured idiosyncratic labor income risks. Among others, Krusell and Smith (1998), Chang and Kim (2007), and Krusell et al. (2012) belong to this class of models. These models can successfully explain the disparity in wealth or income. However, this literature is generally silent about model predictions regarding consumption inequality. Re-examining the leading models mentioned above, I find that they generate only small consumption inequality compared to the data.

In addition, most heterogeneous agent models do not take into account the dynamics of inequality at the business cycle frequencies. Over the business cycles, the second moment aspects of the economy such as inequality fluctuate over time, as well as its first moment aspects such as GDP. As documented in the next section, inequality in the United States is volatile and moderately counter-cyclical. However, as one of the results in this paper shows, standard business cycle models with only shocks to total factor productivity (TFP) fail to explain the cyclicality of inequality.

The purpose of the paper is to develop a business cycle model capable of explaining the levels and dynamics of inequality in earnings, income, and consumption. By investigating the movements of inequality, we may uncover promising driving forces or mechanisms understanding aggregate fluctuations. More importantly, such a model will allow a careful re-examination of the welfare costs of business cycles. In representative agent models, the welfare costs are incredibly small (Lucas, 1987). Heterogeneous household models may provide different insights on the welfare costs. Krusell et al. (2009) measure the welfare costs with heterogeneous agent models, in which the costs are about an order of magnitude larger than that measured by Lucas. Their model, however, does not adequately reproduce the distribution of consumption and the dynamics of inequality, both of which are potentially important in computing such welfare costs. Furthermore, the model I propose may be useful in calculating the welfare costs at disaggregate levels so that we can identify what types of households are the most vulnerable to recessions, which may have implications on effective stabilization policies. Towards these objectives, it is important to have a model that reproduces observed heterogeneity in terms of welfare and its dynamics in different groups of
households.

Before building the model, I use Heathcote, Perri and Violante’s (2010) data to document the stylized facts on inequality. Heathcote et al. (2010) compile a variety of measures for inequality from the Current Population Survey (CPS), the Panel Study of Income Dynamics (PSID), and the Consumer Expenditure Survey (CEX). Among them, the household-equivalent Gini coefficients for earnings, income, and consumption are considered for the analysis in this paper. The Gini coefficient is most widely accepted among the measures of inequality. All of the earnings, income, and consumption are flow variables so that their dynamics are suitable to be considered in the context of business cycles. Among the data set, the measurements from the CPS are used for earnings and income, and the ones from the CEX are used for consumption. Examining the Gini coefficients for earnings, income, and consumption, I find that consumption inequality is significant and inequality is moderately counter-cyclical at an annual frequency.

To explain the levels and dynamics of inequality, a dynamic stochastic general equilibrium model is constructed. In the model economy, households are heterogeneous so that inequality can be addressed (Krusell and Smith, 1998). The households make an intertemporal decision only through the incomplete capital market due to a borrowing constraint (Aiyagari, 1994; Huggett, 1993; Imrohoroğlu, 1989). The source of the heterogeneity is idiosyncratic labor productivity, which causes the households to face uninsured idiosyncratic labor income risks (Chang and Kim, 2007; Takahashi, 2012).

The model features three main ingredients. First, household preferences are non-separable between consumption and leisure and eliminate wealth effects on labor supply as in Greenwood, Hercowitz and Huffman (1988; GHH henceforth). Second, the dispersion of idiosyncratic labor productivity is time-varying as in Takahashi (2012). Third, there is a threshold of idiosyncratic labor productivity for employment. Households with productivity less than the threshold are not able to work. The threshold is time-varying so that, as in Krusell and Smith (1998), the unemployment rate moves in the opposite direction with TFP. Among the three elements, the preference specification is closely related to the levels of inequality, while the last two elements are necessary to explain the dynamics of inequality.

The model departs from Chang and Kim (2007) and Takahashi (2012) in two dimensions, though their models and my model share many aspects. Firstly, Chang and Kim (2007), and Takahashi (2012) assume separable preferences between consumption and leisure as a special case of the King, Plosser and Rebelo’s (1988; KPR henceforth) preferences. The separable
preferences imply strong wealth effects on labor supply. However, the news shock literature provides evidence that no wealth effect on labor supply is not far from the reality and the separable preferences rather imply too strong wealth effects (Jaimovich and Rebelo, 2009; Schmitt-Grohé and Uribe, 2008). Monacelli and Perotti (2008) argue that small wealth effects enable business cycle models to explain the comovement of consumption and real wage to government spending shocks. Guvenen (2009) shows that models with the GHH preferences better explain financial market statistics than those with the KPR preferences.

Secondly, in Chang and Kim (2007) and Takahashi (2012), labor is indivisible (Hansen, 1985; Rogerson, 1988). My model replaces the indivisibility with a time-varying threshold of idiosyncratic labor productivity for employment. This gives rise to a different interpretation on the employment status in my model relative to theirs. In Chang and Kim (2007), and Takahashi (2012), those who do not work are considered as nonemployed rather than unemployed, because they voluntarily decide not to work. In my model, however, every household would like to work, but some of them are forced to be out of employment due to low productivity. Therefore, those who do not work in my model are unemployed.

Non-separable utility eliminating wealth effects on labor supply generates the larger heterogeneity in consumption than separable utility implying strong wealth effects. Eliminating wealth effects on labor supply increases the dispersion of earnings. The strong wealth effects generated by the KPR preferences imply that, given labor productivity, the poor would like to work more than the rich. It means that the strong wealth effects mitigate the disparity in the whole budgets for consumption and savings through labor supply decision. Eliminating the wealth effects shut down this mechanism. A lack of wealth effects on labor supply causes earnings to be more dispersed, which leads to the larger disparity in income and consumption compared with preferences with strong wealth effects.

In addition, the non-separability increases consumption inequality through the slowly diminishing marginal utility of consumption. Under the GHH preferences, the marginal utility of consumption decreases with leisure because of the non-separability, while it is determined solely by consumption under separable preferences. Because leisure and earnings are negatively correlated, the marginal utility of consumption increases with rises in earnings. Since earnings and consumption are positively correlated, the marginal utility of consumption slowly diminishes with rises in consumption under the non-separable preferences. Therefore, households with high earnings would like to consume more because of the higher marginal utility of consumption, and consequently consumption inequality is higher under the non-
separable preferences.

The time-varying dispersion of idiosyncratic labor productivity is similar to uncertainty shocks in the literature in the sense that they are second moment shocks. Most uncertainty shock literature focuses on the time-varying dispersion of firms’ productivity. Bloom (2009) provides evidence on the time-varying uncertainty in the different levels of aggregation of firms’ data. Bachmann and Bayer (2011) and Bloom et al. (2012) show that the time-varying uncertainty is a promising driving force of economic fluctuations in general equilibrium frameworks. On the other hand, Takahashi (2012) provides evidence on the time-varying uncertainty of individuals’ labor productivity. He shows that uncertainty shocks on labor productivity explain the low correlation between total hours worked and aggregate labor productivity.

My idiosyncratic labor productivity process with the time-varying dispersion differs from that in Takahashi (2012) in two ways. First, my time-varying dispersion is mean-preserving. Most uncertainty shock literature including Takahashi (2012) assumes the idiosyncratic productivity to follow a log-AR(1) process with shocks from normal distributions. In the specification, the second moment shocks include the first moment effects because the mean of idiosyncratic productivity, which is log-normally distributed, increases with the dispersion of shocks. In my model, the log-AR(1) process is modified so that the first moment effects can be eliminated and the dispersion shocks can be purely second moment shocks.

Second, the timing of realization of the dispersion shocks is different. In the literature, uncertainty shocks are mostly information or news shocks. Agents in model economies learn future distributions in advance and the knowledge affects current decisions. In this paper, they are contemporaneous shocks, which is the reason that the shocks are called dispersion shocks rather than uncertainty shocks. The contemporaneous shocks are assumed mostly for technical reasons. Given the utility specification in my model and the modified process for idiosyncratic labor productivity, uncertainty shocks in the literature require two state variables: current and future dispersions. The contemporaneous dispersion shocks eliminate the future state and make the model technically more reliable. The technical efficiency is obtained without costs in the sense that my model reproduces the main results in Takahashi (2012).

The time-varying threshold of idiosyncratic labor productivity for employment exogeneously generates the counter-cyclical unemployment rate. In the model, an increase in TFP
decreases the unemployment rate as in Krusell and Smith (1998). The element is necessary for inequality to be moderately counter-cyclical. Because of the negative correlation between TFP and unemployment, TFP shocks make inequality highly counter-cyclical mostly through the extensive margin in the labor market. As discussed later, by contrast, dispersion shocks generate highly pro-cyclical inequality. Therefore, the model has a possibility to explain the cyclicality of inequality depending on the relative sizes of TFP and dispersion shocks.

In computations, the preferences with no wealth effect on labor supply have an advantage over the preferences with non-zero wealth effects. The quantitative analysis is based on the approximate aggregation by Krusell and Smith (1998), which enables us to solve the household problem by replacing the wealth distribution of households with its first moment. The model includes the households’ decision-making for labor supply. The decision requires the information on the real wage. With the preferences implying non-zero wealth effects, calculating the real wage is not trivial because it depends on the exact wealth distribution. This problem can be circumvented by introducing forecasting rules for the real wage or other variables relevant to the labor market [see](Chang and Kim, 2007; Krusell et al., 2012; Takahashi, 2012, for example). In contrast, the preferences eliminating the wealth effects imply that the wealth distribution does not matter for the aggregate labor supply. Therefore, the real wage can be obtained with only the first moment. This makes the wage forecast unnecessary and significantly improves computational efficiency.

The paper is organized as follows. Section 2 documents the stylized facts on inequality, and section 3 presents the model economy. Section 4 discusses the main results, and section 5 examines the role of model ingredients. Finally, section 6 concludes.

2 **Stylized Facts on Inequality**

The special issue “Cross-sectional facts for macroeconomists” of the Review of Economic Dynamics provides systematic analyses on inequality across several countries. In the issue, Heathcote et al. (2010) address a variety of aspects of inequality in the U.S. Unfortunately, they do not formally report facts on inequality in the context of business cycles. Using their data, I document stylized facts on the levels and dynamics of inequality at the business cycle frequencies. Among the several measures of inequality, the household-equivalent Gini

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1 The counter-cyclical unemployment may be endogenized with set-up costs of labor (Rogerson and Wallenius, 2009). I defer incorporating this element to the future work.

2 Similar work is found in Maestri and Roventini (2012).
coefficients for earnings, income, and consumption are considered for the analysis.

Heathcote et al. (2010) use the CEX to measure consumption inequality. The CEX is the only data set in the U.S. that provides detailed information on household consumption expenditures. To measure earnings and income inequality, they use different data sets: the CPS, the PSID, and the CEX. Among them, the measurements based on the CPS are used in this paper. The CEX is not designed to measure accurate earnings and income of households (Cutler and Katz, 1991). The CPS includes richer information than the PSID with respect to the sample size and period. For more details on the data sets and the methods they use to measure inequality, see Heathcote et al. (2010).

The data are annual. The sample period is 25 years from 1980 to 2004, which is the longest period that is covered by all the Gini coefficients for earnings, income, and consumption. Output data are obtained from the National Income and Product Account (NIPA) table. For cyclical components, the natural logarithms of output and the Gini coefficients are detrended by the Hodrick and Prescott’s (1997; HP henceforth) filter with the smoothness parameter of 100.3

2.1 Consumption Inequality Is Significant

Panel (a) of Figure 1 plots the Gini coefficients for earnings, income, and consumption. Earnings inequality is the most severe, income inequality is less serious than earnings inequality, and consumption inequality is the least critical among them without any exceptional year. As reported in Table 1, the average Gini coefficients for earnings, income, and consumption over the sample period are 0.3972, 0.3463, and 0.2994, respectively. Although consumption inequality is the smallest among them, it is not negligible.

Standard economic theories predict that, given that earnings and income include temporary factors, inequality in earnings and income is greater than consumption inequality because individuals tend to smooth their consumption over time. The data confirms the theoretical prediction.

The Gini coefficients in panel (a) of Figure 1 show an increasing trend in inequality over

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3It is not obvious whether the level or logarithm of Gini coefficient should be detrended. One might argue that it should be the level because the Gini coefficient is between zero and one as the unemployment rate and the real interest rate are conventionally detrended without taking a logarithm. However, the Gini coefficient is distinguished from them by its unit. The unit of unemployment rate and real interest rate is percent. One advantage of detrending the logarithm of the Gini coefficient is that the unit of cyclical components becomes consistent between output and the Gini coefficient so that they are directly comparable. Even if the level of the Gini coefficient is used for the analysis, the results in the paper are intact.
the 25 years. Therefore, it would be more implicative to focus on the relative sizes among inequalities. Panel (b) of Figure 1 shows the Gini coefficients for income and consumption relative to the one for earnings. Income inequality has more steeply risen than earnings inequality over 25 years, while consumption inequality has more slowly increased than earnings inequality over the same periods. Although consumption inequality becomes less severe than earnings inequality, it is not obvious for the decreasing trend to be continued because the trend is recently overturned. As tabulated in Table 2, the Gini coefficient of consumption was at least 69.69 percent of earnings inequality, and on average it is 75.45 percent. The Gini coefficient of income was between 83.28 and 91.66 percent of earnings inequality, and on average it is 87.13 percent.

2.2 Inequality Is Moderately Counter-Cyclical

Figure 2 plots detrended Gini coefficients along with cyclical components of output. As shown in the figure, all the inequalities are significantly volatile to consider their cyclical patterns. Earnings inequality is less volatile than output, income inequality is as volatile as output, and consumption inequality is more volatile than output. Table 3 confirms the fact by reporting that the ratios of the standard deviation of Gini coefficient for earnings, income, and consumption to that of output are 0.5607, 0.9941, and 1.1233, respectively.

One interesting feature is that the volatility of consumption inequality is greater than the volatility of earnings and income inequality. One of the reasons is that the unit of cyclical component is the percentage deviation from the trend component. Because the level of consumption inequality is lowest, similar absolute deviations from the trend would cause the volatility of consumption inequality to be higher than the others. This is the case for the volatilities between income and consumption inequality. If the levels of the Gini coefficients are HP-filtered without taking the logarithms, then the standard deviations of the Gini coefficients for earnings, income, and consumption are 0.0041, 0.0063, and 0.0062, respectively. The volatilities of income and consumption are similar.

Figure 2 also shows the cyclical patterns of inequality by comparing the cyclical components of the Gini coefficients with output cycles. Earnings inequality is counter-cyclical over the whole sample period. Income inequality is also counter-cyclical but less than earnings inequality. Consumption inequality was counter-cyclical in the early 1980’s, but it tends to be pro-cyclical in the recent periods. Table 3 describes that the correlations of the Gini coefficient with output are -0.5795, -0.3421, and -0.2188 for earnings, income, and consumption,
respectively.

As shown later, general equilibrium models with only shocks to TFP tend to generate a counterfactually extreme counter-cyclical inequality. The observation in the data might indicate that there are the multiple driving forces of business cycles. Because inequality is the second moment aspects of economy, dispersion shocks or second moment shocks can be a natural choice for another driving force to be considered.

3 Model

A dynamic stochastic general equilibrium model is built to address the relationship between inequalities and business cycles. The economy is populated with a continuum of heterogeneous households and a continuum of identical firms. The households live forever and the firms are static. The households are characterized by their beginning-of-period asset holdings $a_t$ and idiosyncratic labor productivity $x_t$. They have preferences over consumption and leisure. The households allocate time $h_t$ to market work to earn labor income among their time endowment $\bar{h}$. Also they provide capital services to earn capital income. The firms hire both capital and labor services to produce the final good. The prices for capital and labor, $r_t$ and $w_t$, respectively, are competitively determined to clear each market. The households make decisions for consumption $c_t$ and future asset holdings $a_{t+1}$. The asset market is incomplete because the future asset holdings must be greater than or equal to the borrowing constraint $\bar{a}$.

The economy is subject to two exogenous aggregate processes: the one for TFP $z_t$ and the other for the dispersion of idiosyncratic labor productivity $s_t$. While TFP shocks are the first moment shocks, dispersion shocks are the second moment shocks.

Households with labor productivity less than a threshold $\bar{x}_t$ are not able to provide their labor. The threshold of idiosyncratic labor productivity for employment is time-varying so that the unemployment rate has the perfect negative correlation with TFP.

In the subsection, household and firm problems are described and the equilibrium is defined.

3.1 The Heterogeneous Households

The households maximize their life-time expected utility by making their decisions for consumption, hours worked, and end-of-period asset holdings given the factor prices, and other
constraints. The household value function is:

\[ V(a, x; \Omega \equiv \{\mu, s, z\}) = \max_{\{c, h, a, s\}} \left\{ u(c, h) + \beta \mathbb{E}_{x', s', x} V(a', x'; \Omega') \right\}, \tag{1} \]

subject to

\[ c + a' = w(\Omega) x H(x \geq \bar{x}(s, z)) + (1 + r(\Omega)) a \]

\[ \mu' = \Gamma(\Omega, s') \]

\[ a' \in [\bar{a}, \infty) \]

\[ c \in (0, \infty) \]

\[ h \in [0, \bar{h}] \]

where \( \Gamma \) is a transition operator of the household distribution \( \mu \), and \( H(x \geq \bar{x}(s, z)) \) is an indicator function: it is one if \( x \geq \bar{x}(s, z) \), and zero otherwise. The letters with prime indicate the future values of corresponding variables.

### 3.1.1 Period Utility

As Greenwood et al. (1988), the period utility function is assumed as follows:

\[ u(c, h) = \log \left( c - \psi h^{1+\gamma} \right). \tag{2} \]

The preferences are non-separable between consumption and leisure. More importantly, the utility specification eliminates wealth effects on labor supply. As shown shortly, the effective real wage determines the hours worked of households, while their asset holdings play no role in the labor supply decision.

### 3.1.2 Idiosyncratic Labor Productivity

The idiosyncratic labor productivity follows a modified version of log-AR(1) process:

\[ \log(x') = \rho_x \left( \frac{s'}{s} \right) \log(x) + \frac{1}{2} \left( \frac{\rho_x s's' - (s')^2}{1 - \rho_x^2} \right) + s' \epsilon'_x, \quad \epsilon'_x \sim N(0, 1). \tag{3} \]

In the process above, the standard deviation of shocks to labor productivity is time-varying.
Proposition 1 Assume that the idiosyncratic labor productivity follows the process (3). Given the persistence parameter $\rho_x$, the current dispersion state $s$ solely determines the current distribution of idiosyncratic labor productivity as follows:

$$x \sim LN\left(\frac{-\frac{1}{2}s^2}{1-\rho_x^2}, \frac{s^2}{1-\rho_x^2}\right).$$

Furthermore, the mean of idiosyncratic labor productivity is unity regardless of the dispersion state.

Proof. See Appendix A.1.

The ratio of future to current dispersion states ($s'/s$) in the first term of the process is included so that the productivity distribution may move to the limit distribution without delay whenever the dispersion state changes. Without the term ($s'/s$) as in a typical AR(1) process, upon a shift in the dispersion state, the productivity distribution moves slowly toward its new limit distribution. In that case, to know the current distribution of labor productivity, the necessary is the information on the lagged distribution. This characteristic is useful because the current dispersion state reveals the current distribution of labor productivity. Under the preferences without the wealth effect on labor supply, the productivity distribution with aggregate states directly determines the aggregate labor supply.

The second term in the process is to adjust the mean of labor productivity when dispersion state changes. Without the term, the log-normal distribution implies that the mean of labor productivity is always greater than unity and an increase in productivity dispersion raises the mean of labor productivity. In other words, although the dispersion shocks are the second moment shocks, they have the first moment effects. The second term eliminates the first moment effects of the second moment shocks.

3.1.3 The Dispersion of Idiosyncratic Labor Productivity

The standard deviation of idiosyncratic labor productivity follows a log-AR(1) process with an unconditional mean $\bar{s}$:

$$\log(s') = (1 - \rho_s) \log(\bar{s}) + \rho_s \log(s) + \sigma_s \epsilon'_s, \quad \epsilon'_s \sim N(0, 1).$$

(4)

As aggregate states indicate, the dispersion state is realized at the beginning of the period, while it is commonly revealed in a period ahead in the literature.
3.1.4 Total Factor Productivity

The TFP follows a log-AR(1) process:

\[
\log(z') = \rho_z \log(z) + \sigma_z \epsilon_z', \quad \epsilon_z' \sim N(0, 1).
\]

All the shocks \((\epsilon_x, \epsilon_s, \epsilon_z)\) in the model are mutually independent.

3.1.5 Labor Supply

From the household problem, given the market wage \(w\) per effective unit of labor supply, the optimality condition for the interior solution of hours worked is

\[
\psi \tilde{h}^\gamma = wx.
\]

The condition implies that households decide their hours worked, if the hours are not bound, by the following rule:

\[
\tilde{h}(x; w) = \left(\frac{wx}{\psi}\right)^\frac{1}{\gamma}.
\]

By taking into account the time endowment and the threshold of idiosyncratic labor productivity for employment, the policy function for hours worked of households is

\[
H(x; \Omega) = h(x; w(\Omega)) = \begin{cases} 
\tilde{h} & \text{if } x > \hat{x}(w) \\
\left(\frac{wx}{\psi}\right)^\frac{1}{\gamma} & \text{if } \bar{x}(s, z) \leq x \leq \hat{x}(w) \\
0 & \text{if } x < \bar{x}(s, z)
\end{cases}
\]

where \(\hat{x}(w)\) is the upper bound of labor productivity for the interior solution and is derived as

\[
\hat{x}(w) = \frac{\psi}{w} \tilde{h}^\gamma.
\]

The policy function for hours worked confirms the fact that the household labor supply decision only depends on the effective wage. In other words, there is no wealth effect on labor supply.
Proposition 2 Assume that households make their decisions for hours worked according to (5) given a time endowment \( \bar{h} \), and that idiosyncratic labor productivity \( x \) follows the process (3). Then, given TFP \( z \) and a dispersion state \( s \), the aggregate labor supply depends only on the market real wage. In particular, the aggregate labor supply schedule is

\[
L^S(w; s, z) = \left( \frac{w}{\psi} \right)^{\frac{1}{\gamma}} A(w; s, z) + \bar{h} B(w; s),
\]

where

\[
A(w; s, z) = \int_{\hat{x}(w)}^{\hat{x}(s, z)} x^{1+\frac{1}{\gamma}} f(x; s) dx, \quad \text{and} \quad B(w; s) = \int_{\hat{x}(w)}^{\infty} x f(x; s) dx,
\]

and \( f(x; s) \) is a probability density function of idiosyncratic labor productivity.

Proof. See Appendix A.2.

Proposition 2 shows an important feature of the preference specification. Because the preferences eliminate the wealth effect on labor supply, the labor supply only depends on exogenous aggregate states \((s, z)\) given the real wage, but not on the wealth distribution of households. As shown shortly, this characteristic makes the quantitative analysis efficient because it enables households to learn the exact equilibrium real wage and labor before their decision-making although they have access to only a small piece of information on the wealth distribution.

Remark 1 In Proposition 2, \( L^S(w; s, z) \) is increasing in the market real wage \( w \).

Proof. See Appendix A.3.

In Proposition 2, it is ambiguous if the labor supply schedule is upward sloping. The upper bound of interior solution for hours worked \( \hat{x}(w) \) is decreasing in the real wage. When the real wage rises, therefore, the function \( A(w; s, z) \) decreases because of the decline in the upper limit of integral, while \( B(w; s) \) and \((w/\psi)^{1/\gamma}\) increase. However, it turns out that the labor supply schedule is upward sloping as shown in Remark 1.

Remark 2 Assume that the threshold of idiosyncratic labor productivity \( \bar{x}(s, z) \) is time-varying so that the unemployment rate has the perfect negative correlation with TFP \( z \). Then, in Proposition 2, \( L^S(w; s, z) \) is increasing in TFP.
Proof. See Appendix A.4. \hfill ■

Shocks to TFP have an impact on the labor supply schedule. Remark 2 proves that the labor supply schedule shifts to the right with an increase in TFP.

## 3.2 The Identical Firms

The firms maximize their profits by producing output according to the constant returns to scale Cobb-Douglas technology in capital $K$ and labor $L$ given the real wage and the rental price of capital. A $\delta$ fraction of capital is depreciated during the production. The firms are responsible for the depreciation. The firm’s profit maximization problem is

$$\max_{\{K,L\}} \left\{ zK^{\alpha}L^{1-\alpha} - w(\Omega)L - \left( r(\Omega) + \delta \right) K \right\}.$$  

From the firm’s problem, we have the following labor demand schedule:

$$L_D(w; K, z) = \left(1 - \alpha\right) \frac{z}{w} K^{\frac{1}{\alpha}}.$$  \hfill (7)

The aggregate capital $K$ is a predetermined variable by the construction of the model. The labor demand schedule is downward sloping in the real wage. An increase in the aggregate capital or TFP shifts out the labor demand schedule from the origin. The dispersion shocks play no role in the labor demand.

## 3.3 Recursive Competitive Equilibrium

A recursive competitive equilibrium is a set of functions,

$$\{V, C, A, H, K, L, r, w, \Gamma\},$$

that solve household and firm problems and clear markets for output, capital, and labor as described by the following conditions.

1. $V(a, x; \Omega)$ is the value that solve household problem, and $\{C(a, x; \Omega), A(a, x; \Omega), H(x; \Omega)\}$ are the associated policy functions for consumption, future asset holdings, and labor supply, respectively.

2. $\{K(\Omega), L(\Omega)\}$ are the policy functions that maximize firm’s profits.
3. Prices are competitively determined:

\[ r(\Omega) = \alpha z \left( \frac{K(\Omega)}{L(\Omega)} \right)^{\alpha-1} - \delta \] (8)

\[ w(\Omega) = (1 - \alpha) z \left( \frac{K(\Omega)}{L(\Omega)} \right)^\alpha . \] (9)

4. The goods market clears:

\[ zK(\Omega)^\alpha L(\Omega)^{1-\alpha} + (1 - \delta)K(\Omega) = \int C(a, x; \Omega) \mu( d[a \times x]) + \int A(a, x; \Omega) \mu( d[a \times x]) . \]

5. The capital market clears:

\[ K(\Omega) = \int a \mu( d[a \times x]) . \]

6. The labor market clears:

\[ L(\Omega) = \int xH(x; \Omega) \mu( d[a \times x]) . \]

7. Given the exogenous processes (3) and (4), the transition operator of household distribution \( \Gamma(\Omega, s') \) is consistent with household decision rules: For all \( A \subset [\bar{a}, \infty) \) and \( X \subset (0, \infty) \),

\[
\mu'(A, X') = \int \left\{ \left( a, x \right) \mid A(a, x; \Omega) \in A \right\} \int_{s' \in (0, \infty)} \int_{x' \in X} P_x(dx'|x, s, s') P_s(ds'|s) \mu( d[a \times x]) ,
\]

where \( P_x(x'|x, s, s') \) and \( P_s(s'|s) \) are the conditional distributions of \( x' \) and \( s' \), respectively.

4 Quantitative Results from Benchmark Model

4.1 Parameterization for Benchmark Model

Most of the parameters are comparably set to the literature on the incomplete asset market models with heterogeneous households and idiosyncratic labor income risks (Chang and Kim, 2007; Krusell and Smith, 1998; Takahashi, 2012). The model period corresponds to one
quarter. The time endowment $\bar{h}$ is normalized to 1. For the preference parameters, the subjective discount factor is set to 0.9878 to match the average quarterly real interest rate of 1 percent. The elasticity of labor supply $1/\gamma$ is assumed to be 1. The disutility parameter $\psi$ is chosen to be 7.3712, which matches the average labor input of $1/3$ conditional on working. For the production technology, the capital share of output $\alpha$ is 0.36, and the depreciation rate of capital $\delta$ is 0.025.

The borrowing constraint $\bar{a}$ is assumed to be 0 because the model is well defined only over non-negative borrowing constraints. If households are allowed to hold debts (negative assets), in the model economy, the unemployed whose debts are close to the borrowing constraint cannot repay their debts and cannot survive because positive consumption is not feasible. Non-negative borrowing constraints remove the case.

There are three exogenous processes. All the processes are discretized with Rouwenhorst’s (1995) method. The number of states for TFP and dispersion is 3 for both. The number of idiosyncratic labor productivity is 15. $\rho_z$ is chosen to be 0.8257 to match the average length of cycles of 24 quarters in the postwar U.S. data based on the business cycle dates by National Bureau of Economic Research. $\rho_s$ is also set to 0.8257 to match the average length of risk cycles of 24 quarters by Takahashi (2012). $\rho_x$ is set to 0.915 following Takahashi (2012). $\sigma_x$ is chosen to be 0.028 to explain the average annual standard deviation of aggregate risks of 4.4 percent as in Takahashi (2012). Given $\sigma_s$, $\sigma_z$ is set at 0.008 to explain the average volatility of output. Finally, $\bar{s}$ is set at 0.15 to match the average Gini coefficient of earnings of 0.3927. Table 4 lists the parameter values.

One of the important components in the model is the time-varying threshold of idiosyncratic labor productivity for employment. The thresholds are chosen to roughly match the movement of unemployment rate in the U.S. data (Krusell and Smith, 1998). As shown later, although the model has two aggregate shocks, TFP mostly drives the economy. As mentioned above, TFP has three states: high, medium, and low TFP. Accordingly, the unemployment rate is also realized at one of three values: 0.287, 0.592, and 0.898. If current TFP is high (medium or low), the productivity threshold is chosen to match the low (medium or high) unemployment rate of 0.287 (0.592 or 0.898) given the dispersion state. This exogenously generates counter-cyclical unemployment.
4.2 Business Cycle Statistics

Before reporting statistics on inequality, it is important to see if the model economy is successful in explaining typical business cycle moments. Along with the dynamics of consumption and investment, the particularly interested are statistics on labor market. The model includes the time-varying dispersion of idiosyncratic labor productivity, which is second moment shocks. A similar ingredient is found in Takahashi (2012). In his model, the second moment shocks are the key to explain the labor market statistics, especially the low correlation of total hours and aggregate labor productivity in the data.

Table 5 summarizes the statistics. To have cyclical components, the logarithms of simulation data are HP-filtered.\textsuperscript{4} The benchmark model generates the similar magnitude of output fluctuations. The model successfully replicates the dynamics of consumption and investment in the data. Both in the data and model, consumption is less volatile and investment is more volatile than output. The model also explains the highly pro-cyclical consumption and investment as in the data.

The model economy also replicates some of the features observed in the data on the labor market. Total hours are roughly as much volatile as output, and are highly pro-cyclical. Aggregate labor productivity is less volatile than output, and is moderately pro-cyclical. In particular, the model explains the low correlation between total hours and aggregate labor productivity.

As Takahashi (2012), the critical element of the model is the time-varying dispersion of idiosyncratic labor productivity to explain the labor market statistics. Although the role of dispersion shocks is discussed in more detail in the next section, it would be good to briefly mention why the time-varying dispersion is necessary to take into account the movements in the labor market. In the literature, standard business cycle models only with TFP shocks commonly face the problem of counterfactually high correlation between total hours and aggregate labor productivity. This is mainly because, in standard general equilibrium models, labor supply schedules do not much shift with TFP shocks, and so the labor market is mostly subject to its demand side. In my benchmark model, in contrast, dispersion shocks enable the supply side to also play an important role in the labor market. Consequently, the supply and demand schedules interact each other in the environment with uncorrelated TFP and

\textsuperscript{4}In Table 5, the data frequency of business cycle statistics is quarterly, but the one of inequality statistics is annul. Accordingly, 1,600 and 100 of smoothness parameter for the HP-filter are used for business cycle and inequality statistics, respectively. $\sigma_Y$ is multiplied by 100 so the unit of it is the percentage deviation from its trend.
dispersion shocks, which reduces the correlation of total hours and aggregate productivity.

4.3 Inequality

The benchmark model well explains the order of the Gini coefficients\(^5\) for earnings, income, and consumption. Both in the data and model, earnings inequality is the highest, consumption inequality is the lowest, and income inequality is between the two. In the model, the Gini coefficients for income and consumption are 92.01 percent and 65.44 percent, respectively, of the Gini coefficient of earnings, while they are 87.13 percent and 75.45 percent, respectively, in the data. In other words, income inequality is slightly overstated, but consumption inequality is understated compared with earnings inequality. Though consumption inequality is undervalued, the model improves on the heterogeneity of consumption with non-separable preferences eliminating the wealth effect on labor supply compared with separable preferences with a strong wealth effect. The relationship between preferences and inequality is addressed in more detail in the next section.

In the model, all the inequalities have low correlations with output. The correlations with output of inequality for earnings, income, and consumption are -0.1049, -0.1730, and 0.0261, respectively, in the model. As discussed below, TFP shocks and dispersion shocks have an opposite implication on the cyclicality of inequality. TFP shocks, which are negatively correlated with the unemployment rate, make inequality highly counter-cyclical. Dispersion shocks, on the other hand, generate highly pro-cyclical inequality. Therefore, a model with a single type of aggregate shocks fails to explain moderately counter-cyclical inequality.

A drawback of the model comes with the volatility of inequality. In the data, earnings inequality is less volatile than output, income inequality is as much volatile as output, and consumption inequality is more volatile than output. In the model, however, all the inequalities are more volatile than output. More seriously, the model generates the opposite order of volatilities among the three inequalities. This result seems natural under the environment of standard macroeconomic models. Households would like to smooth their consumption over time so that consumption inequality is likely to be less volatile than earnings and income

\(^5\)As Krusell et al. (2009), the Gini coefficient is calculated by the following formula:

\[
G_f = \frac{1}{n} \left[ n + 1 - 2 \left( \frac{\sum_{i=1}^{n} (n + 1 - i) f_i}{\sum_{i=1}^{n} f_i} \right) \right],
\]

where \(n\) is the population of the economy and \(f_i\) can be the earnings, income, or consumption of household \(i\), which is re-arranged in a non-decreasing order, i.e., \(f_j \leq f_{j+1}\) for all \(j\).
inequality. Therefore, to explain the volatility of inequality, the model needs another ingredient that prevents some of the households from smoothing their consumption. This may be a future research.

5 The Role of Model Ingredients

In this section, the roles of model ingredients are discussed. Firstly, using the Jaimovich and Rebelo’s (2009; JR henceforth) preferences, it is addressed how wealth effects on labor supply affect the levels of inequality. Secondly, the roles of TFP and dispersion shocks are investigated. Third, it is shown that the unemployment rate which is negatively correlated with TFP is necessary to account for the cyclicality of inequality. Finally, it is examined how different parameterizations for the Frisch elasticity of labor supply affect model statistics.

5.1 Inequality and Wealth Effects on Labor Supply

One of the main ingredients in the model is the non-separable utility eliminating wealth effects on labor supply as the GHH preferences. In the literature (Chang and Kim, 2007; Krusell et al., 2012; Takahashi, 2012, for example), by contrast, preferences with strong wealth effects as the KPR preferences are widely adopted. However, given the heterogeneity of earnings or income, the large disparity in consumption observed in the data is hardly reproduced under preferences with strong wealth effects.

For the quantitative analysis on the role of preferences in consumption inequality, a type of the JR preferences is considered. The period utility function is

$$ u(c, h; m_{-1}) = \log \left( c - \psi \frac{h^{1+\gamma}}{1+\gamma} m \right) $$

$$ m = c^\theta m_{-1}^{1-\theta}, $$

where $m_{-1}$ is a predetermined variable. The parameter $\theta$, which is in $[0, 1]$, governs wealth effects on labor supply (Jaimovich, 2008). In particular, $\theta = 1$ gives rise to separable preferences between consumption and leisure with strong wealth effects. As $\theta \to 0$, the utility function approaches to the benchmark utility, i.e., non-separable preferences eliminating wealth effects on labor supply.

The stationary analysis is conducted to see the relationship between wealth effects on
labor supply and the levels of inequality.\textsuperscript{6} Table 9 and Figure 3 summarize the results. As shown in panel (a) of the figure, all the Gini coefficients decrease with an increase in $\theta$. The rich with the strong wealth effects have less incentive to supply labor than the poor. Therefore, the wealth effects mitigate earnings inequality and consequently income and consumption inequality. Under the preferences eliminating wealth effects on labor supply ($\theta = 0$), the Gini coefficients for earnings, income, and consumption are 0.4108, 0.3784, and 0.2660, respectively, while they are 0.2954, 0.2699, and 0.1348, respectively, with separable preferences with strong wealth effects ($\theta = 1$).\textsuperscript{7}

More importantly, panel (b) of the figure shows that preferences with strong wealth effects fail to generate enough consumption inequality given earnings inequality. The ratio of consumption inequality to earnings inequality is 0.7545 in the U.S. data. In the figure, the ratio monotonically decreases from 0.6475 to 0.4564 as $\theta$ approaches from 0 to 1. On the other hand, the wealth effects has essentially no effect on the ratio of income inequality to earnings inequality. Given earnings or income inequality, therefore, strong wealth effects on labor supply dampen the heterogeneity of consumption across households.

Under separable preferences, households require only positive consumption. Under the benchmark preferences, on the other hand, the disutility of labor supply must be compensated by consumption. In other words, the disutility plays a role as the lower bound of consumption. Therefore, households with higher earnings due to supplying more hours worked would like to consume more, which causes higher inequality of consumption compared with separable preferences. Furthermore, under non-separable preferences, the disutility of labor supply prevents the marginal utility of consumption from quickly diminishing. In other words, a more time allocation to working increases earnings, and given consumption, it also increases the marginal utility of consumption. Households with high earnings would like to consume more because of the high marginal utility compared with separable preferences, in which the marginal utility of consumption depends only on consumption. This channel also amplifies the heterogeneity of consumption.

\textsuperscript{6}Given $\theta$, $\psi$ is chosen so that average hours worked is 1/3 conditional on working. 

\textsuperscript{7}When $\theta = 0$, the Gini coefficients are higher than those from the benchmark model. The reason is that, in the dynamic model, the annual Gini coefficients are reported from the quarterly model by aggregating four consecutive data to have annual data. In other words, in the simulation data, the first through the fourth period form the first year, the fifth to the eighth period form the second year, and so on. The time-aggregation slightly reduces inequality. In the stationary analysis, however, the time-aggregate is impossible.
5.2 TFP and Dispersion Shocks

In the model, business cycles are driven by two aggregate shocks: TFP and dispersion shocks. It is useful to see how these shocks play a role in the model economy. Table 6 reports the business cycle and inequality statistics. In the table, columns named ‘both’ copy the results from the benchmark model. The z-shock model keeps TFP shocks but eliminates dispersion shocks from the benchmark model, while the s-shock model holds dispersion shocks but removes TFP shocks.

Both types of aggregate shocks have an influence on the economy mainly through labor market. As shown in Table 6, the labor market statistics between the z-shock and s-shock model are quite different, while the dynamics of consumption and investment are similar. In the model, both shocks are necessary to explain the dynamics of labor market. Furthermore, both shocks play an important role in the dynamics of inequality because earnings inequality is closely related to the labor market movements, and thereby the shocks affect income and consumption inequality.

TFP and dispersion shocks have an impact on the labor market in two different ways. Firstly, TFP shocks affect the labor market through both the intensive and extensive margin, whereas dispersion shocks influence it only through the intensive margin. In the model, the unemployment rate has the perfect negative correlation with TFP. The extensive margin channel makes total hours more volatile in the z-shock model. Table 6 confirms the fact by showing that $\sigma_H/\sigma_Y$’s in the z-shock and s-shock model are 0.8841 and 0.4130, respectively. This is also demonstrated with the comparison of panel (a) and (c) of Figure 4. In panel (a), the aggregate hour schedule significantly shifts to the right when TFP increases. In panel (c), on the other hand, dispersion shocks negligibly affect the aggregate hour schedule.

Secondly, as displayed in panel (b) and (d) of Figure 4, TFP shocks drive the economy with labor demand, while dispersion shocks do with labor supply. Although positive TFP shocks allow more households to supply labor, their productivity is so low that the rise in effective labor supply is negligible. In contrast, an increase in TFP causes firms to hire more labor inputs to take advantage of higher productivity so that the labor demand schedule significantly shifts out. Therefore, positive TFP shocks raise both the equilibrium labor and real wage. The rise in labor leads output to increase, and the rise in the real wage with the shift of total hour schedule in panel (a) causes the total hours to increase too. Consequently, TFP shocks lead output and total hours to move in the same directions as Table 6 reports that in the z-shock model, $\rho(Y, H) = 0.9991$. 
In the z-shock model, output and total hours move together, and output is more volatile than total hours. Therefore, aggregate labor productivity is counterfactually smoothly moving, and it has a too high and positive correlation both with output and total hours.

When it comes to dispersion shocks, they have no influence on labor demand at all as displayed in panel (d) of Figure 4, but they shift the labor supply schedule to the right. The increase in labor supply is mainly due to the increase in the average idiosyncratic labor productivity of working households. Dispersion shocks asymmetrically affect the households. With the dispersion shocks, high productive households become more productive, whereas low productive households become less productive. In the model economy, low productive households are not allowed to work. As a result, the asymmetric impact of dispersion shocks tends to increase the average labor productivity of working households.

As panel (d) implies, an increase in the dispersion of idiosyncratic labor productivity raises the equilibrium labor, which increases output, but cut down the real wage, which decreases total hours. Accordingly, output and total hours tend to move in the opposite directions. This causes aggregate labor productivity to be counterfactually more volatile than output, and to have too high positive and negative correlations with output and total hours, respectively.

Neither z-shock nor s-shock model explains the dynamics of labor market. In particular, the correlation between total hours and aggregate labor productivity is either highly procyclical or highly counter-cyclical in the model only with a single type of aggregate shocks, while the correlation is low in the data.

Incorporating both types of shocks as the benchmark model resolves the problem. Figure 5 describes the responses of labor market to the combinations of TFP and dispersion shocks. All the panels indicate that TFP shocks govern the direction of movement of equilibrium labor and real wage, while dispersion shocks adjust the relative sizes of response between the labor and real wage. When TFP increases as in panel (a) and (b), both the equilibrium labor and real wage increase so that output and total hours rise. Table 6 confirms the fact by reporting that $\rho(Y, H) = 0.9088$ and $\sigma_H/\sigma_Y = 0.8728$ in the benchmark model, which are closer to the model statistics in the z-shock model.

In panel (a) of Figure 5, a rise in the dispersion of idiosyncratic labor productivity amplifies the response of labor but dampens the movement of real wage. In that case, output moves more than total hours. As a result, aggregate labor productivity tends to move together with output and total hours. On the other hand, in panel (b), a decline in the dispersion
dampens labor movements and amplifies real wage responses so that the responses of total hours dominate output movements. Consequently, aggregate labor productivity moves in the opposite directions with output and total hours. In the benchmark model, TFP shocks and dispersions shocks are mutually independent. The mixed shocks result in the low correlation of labor productivity both with output. Furthermore, the relative volatility of aggregate labor productivity is also closer to that in the data.

As for inequality, in general, aggregate shocks play an important role in the dynamics rather than in the levels. All the models in Table 6 report almost the same levels of the Gini coefficients for earnings, income, and consumption. However, the model statistics on the dynamics are quite different. Inequalities are smoother in the z-shock model than in the s-shock model, and the volatilities of the Gini coefficients in the benchmark model are between the two models with a single type of shocks. Obviously, the time-varying dispersion of idiosyncratic labor productivity amplifies the movements of earnings inequality, and thereby income and consumption inequality. Besides, the high volatility of inequality in the s-shock model is partly due to the low volatility of output.

TFP and dispersion shocks also have different implications on the cyclicality of inequality. In the z-shock model, inequality is highly counter-cyclical. This is mainly due to the extensive margin in the labor market. Higher TFP allows some of the low productive households to work and earn labor income. Accordingly, earnings inequality, as well as income and consumption inequality, is reduced and counter-cyclical. In contrast, dispersion shocks only affect the intensive margin. Higher dispersion implies that incumbent working households, on average, become more productive, and so they provide more hours and effective labor supply. Consequently, an increase in dispersion of idiosyncratic labor productivity causes more severe inequality in earnings, and thereby income and consumption inequality. Since a rise in the dispersion increases output, inequality becomes pro-cyclical in the s-shock model. In the benchmark model, uncorrelated TFP and dispersion shocks keep hitting the economy so that the cyclicality of inequality is reduced.

5.3 The Time-Varying Threshold of Idiosyncratic Labor Productivity for Employment

As in Krusell and Smith (1998), unemployment has the perfect negative correlation with TFP. Table 7 compares the model statistics between from the benchmark model (column ‘unemployment’) and from the model with full employment. Given the same distribution of
labor productivity, completely eliminating the threshold causes earnings distribution to be more concentrated since low productivity households are allowed to supply their labor and catch up the earnings of other groups of households. In the full employment model, therefore, $\bar{s}$ is re-calibrated to target the same Gini coefficient for earnings as the benchmark model. In the full employment model, $\bar{s} = 0.156$, which is greater than 0.015 in the benchmark model. Accordingly, $\psi$ and $\beta$ are reset to 7.1217 and 0.9879, respectively, for both models to target the same average hours worked of working households and quarterly real interest rate.

As for business cycle statistics, the threshold has a negligible impact on the dynamics of consumption and investment. Obviously, however, it significantly affects the statistics on labor market. Time-varying unemployment makes total hours worked more volatile through the extensive margin so that the model statistics become closer to that in the data. Aggregate labor productivity becomes less volatile in the benchmark model, though the model statistics from both models are around the data. In expansions with positive TFP shocks, output increases because of the increases in TFP and in employment. However, a rise in TFP also allows households with low labor productivity to supply labor so that the average idiosyncratic labor productivity of working households decreases during expansions. Overall the aggregate labor productivity increases but less volatile in the benchmark model than in the full employment model.

As for the cyclicality of labor market, total hours worked become more important in economic fluctuations in the benchmark model because of the extensive margin. The correlation between output and total hours is higher in the benchmark model. As discussed in the previous paragraph, however, the extensive margin causes the aggregate labor productivity to be less pro-cyclical and so the correlation between total hours and aggregate labor productivity becomes lower in the benchmark model. Overall, the benchmark model better explains the labor market than the full employment model.

The benchmark model, in general, better explains the inequality statistics. Given similar earnings inequality, time-varying unemployment has essentially no impact on the levels of inequality. The benchmark and full employment model, however, imply the different dynamics of inequality. In the benchmark model, the volatilities of the Gini coefficients relative to output volatility are smaller and closer to those in the data. This is partly because output becomes more volatile in the benchmark model. More importantly, time-varying unemployment better explains the cyclicality of inequality compared with the full employment model. Because unemployment is counter-cyclical by construction, inequality becomes more counter-
cyclical in the benchmark model. In the data, inequality is moderately counter-cyclical, while it is moderately pro-cyclical in the full employment model. The cyclicity of inequality in the benchmark model is between the statistics in the data and the full employment model.

5.4 The Frisch Elasticity of Labor Supply: Trade Off

Under the utility specification of benchmark model, the Frisch elasticity of labor supply is $1/\gamma$. The benchmark model assumes the unit elasticity of labor supply. However, there is little consensus about the elasticity, particularly between microeconomic and macroeconomic literature. Empirical literature with micro-data insists that the labor supply is inelastic. In most macroeconomic literature, on the other hand, elastic labor supply is required to explain economic fluctuations.

The models with low ($1/\gamma = 0.5$) and high ($1/\gamma = 1.5$) elasticity of labor supply are investigated. Depending on $\gamma$, $\bar{s}$, $\psi$, and $\beta$ are re-calibrated as discussed in the previous subsection. Table 8 compares the results from three different sets of parameterizations. In general, the business cycle statistics support the low elasticity, while the inequality statistics defend the high elasticity.

The model with the low elasticity of labor supply improves on the business cycle statistics, especially the labor market statistics, compared with the benchmark model (unit elasticity). The low elasticity makes total hours relatively less important in economic fluctuations. The correlation between output and total hours decreases with the low elasticity and becomes closer to that in the data. Due to the low correlation, aggregate labor productivity is more volatile, and the correlations of aggregate labor productivity with output and total hours move toward the negative one. Because of the low elasticity, the volatility of output decreases. As for consumption and investment, consumption becomes less volatile while investment becomes more volatile. Both consumption and investment are less correlated with output.

On the other hand, the model with the high elasticity of labor supply improves on the inequality statistics. Given earnings inequality, the high elasticity increases the heterogeneity of consumption compared to the benchmark model. The overall volatility of inequality relative to output volatility is reduced partly due to the high volatility of output. Finally, the inequality becomes more counter-cyclical in the high elasticity of labor supply.
6 Conclusion

In order to account for the levels and dynamics of inequality with business cycles, a heterogeneous household general equilibrium model with idiosyncratic labor income risks is proposed. Non-separable preferences between consumption and leisure eliminating wealth effects on labor supply increase the average disparity in consumption. The time-varying dispersion and threshold of idiosyncratic labor productivity both are necessary to explain the low correlation between output and inequality. Along with the inequality statistics, the model well replicates the business cycle features observed in the data, particularly on the labor market.

One of the contributions of the paper is to provide a better framework to examine the welfare costs of business cycles than existing models in the literature, in the sense that the model better explains the disparity in welfare and its dynamics. Along with the average welfare costs, heterogeneous household models enable us to investigate the welfare costs in different groups of households. To measure the costs in disaggregate levels, it is important for the model to well reproduce the welfare distribution and its dynamics.\footnote{I am currently trying to use the model to measure the welfare costs of business cycles, which may have implications on effective stabilization policies.}

In the model, unemployment is exogenously generated by the time-varying threshold of idiosyncratic labor productivity for employment. This element might be endogenized by the set-up costs of labor (Rogerson and Wallenius, 2009). A simple example of the set-up costs is the amount of hours that workers need to commute to their work places and coordinate their work. Because of the set-up costs, for some initial hours, labor productivity is essentially zero. If the hours are not paid and the optimal hours worked for a household are less than the set-up costs, then the household would decide not to work. The set-up costs change the concept about employment status. Those who do not work are not the unemployed but the nonemployed.

A drawback of the model is that it hardly explains the order of volatilities among earnings, income, and consumption inequality. In the data, consumption inequality is more volatile than others. The relatively high volatility of consumption inequality might be explained by hand-to-mouth consumers. If consumption smoothing technologies are only available to the rich but not to the poor, consumption inequality would become more volatile than others such as earnings inequality. My model does not include this mechanism, and it can be a future research topic.
References


### Table 1: Descriptive Statistics for Gini Coefficients

<table>
<thead>
<tr>
<th></th>
<th>Earnings</th>
<th>Income</th>
<th>Consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.3972</td>
<td>0.3463</td>
<td>0.2994</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.4231</td>
<td>0.3753</td>
<td>0.3132</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.3641</td>
<td>0.3057</td>
<td>0.2804</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0163</td>
<td>0.0192</td>
<td>0.0075</td>
</tr>
</tbody>
</table>

**Source:** Heathcote et al. (2010)

**Note:** The data are annual. The statistics are calculated over the common sample period from 1980 to 2004. The CPS is used for earnings and income, and the CEX is used for consumption. Consumption includes both nondurables and services.

### Table 2: Relative to Gini Coefficient of Earnings

<table>
<thead>
<tr>
<th></th>
<th>Income</th>
<th>Consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.8713</td>
<td>0.7545</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.9166</td>
<td>0.8190</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.8328</td>
<td>0.6969</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.0179</td>
<td>0.0271</td>
</tr>
</tbody>
</table>

**Source:** Heathcote et al. (2010)

**Note:** The data are annual. The statistics are calculated over the common sample period from 1980 to 2004. The CPS is used for income, and the CEX is used for consumption. Consumption includes both nondurables and services.

### Table 3: Volatility and Cyclical Pattern of Gini Coefficient

<table>
<thead>
<tr>
<th>X:</th>
<th>Earnings</th>
<th>Income</th>
<th>Consumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma(X)/\sigma(Y)$</td>
<td>0.5607</td>
<td>0.9941</td>
<td>1.1233</td>
</tr>
<tr>
<td>$\rho(X,Y)$</td>
<td>-0.5795</td>
<td>-0.3421</td>
<td>-0.2188</td>
</tr>
</tbody>
</table>

**Source:** Heathcote et al. (2010) and NIPA Tables

**Note:** $Y$ is output. The data are annual. The statistics are calculated over the common sample period from 1980 to 2004. The CPS is used for income, and the CEX is used for consumption. Consumption includes both nondurables and services. The logarithms of output and the Gini coefficients are detrended by the HP-filter with the smoothing parameter of 100.
### Table 4: Parameterization for Benchmark Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.9878</td>
<td>subjective discount factor</td>
</tr>
<tr>
<td>$1/\gamma$</td>
<td>1.0</td>
<td>elasticity of labor supply</td>
</tr>
<tr>
<td>$\psi$</td>
<td>7.3712</td>
<td>disutility parameter</td>
</tr>
<tr>
<td>$\bar{h}$</td>
<td>1.0</td>
<td>time endowment</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.36</td>
<td>capital share of output</td>
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<tr>
<td>$\delta$</td>
<td>0.025</td>
<td>depreciation rate</td>
</tr>
<tr>
<td>$\bar{a}$</td>
<td>0.000</td>
<td>borrowing constraint</td>
</tr>
<tr>
<td>$\rho_z$</td>
<td>0.8257</td>
<td>persistence of TFP</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.007</td>
<td>std. dev. of TFP shocks</td>
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<tr>
<td>$\rho_x$</td>
<td>0.915</td>
<td>persistence of labor productivity</td>
</tr>
<tr>
<td>$\bar{s}$</td>
<td>0.15</td>
<td>ave. std. dev. of labor prod. shocks</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>0.8257</td>
<td>persistence of dispersion state</td>
</tr>
<tr>
<td>$\sigma_s$</td>
<td>0.028</td>
<td>std. dev. of dispersion shocks</td>
</tr>
</tbody>
</table>

### Table 5: Benchmark Model Statistics

#### Business Cycles

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_Y$</td>
<td>1.6268</td>
<td>1.6408</td>
<td>$\rho(Y,C)$</td>
<td>0.9242</td>
</tr>
<tr>
<td>$\sigma_C/\sigma_Y$</td>
<td>0.5022</td>
<td>0.5209</td>
<td>$\rho(Y,I)$</td>
<td>0.9681</td>
</tr>
<tr>
<td>$\sigma_I/\sigma_Y$</td>
<td>2.7919</td>
<td>2.4129</td>
<td>$\rho(Y,H)$</td>
<td>0.8234</td>
</tr>
<tr>
<td>$\sigma_H/\sigma_Y$</td>
<td>0.9510</td>
<td>0.8728</td>
<td>$\rho(Y,Y/H)$</td>
<td>0.3730</td>
</tr>
<tr>
<td>$\sigma_{Y/H}/\sigma_Y$</td>
<td>0.5816</td>
<td>0.4189</td>
<td>$\rho(H,Y/H)$</td>
<td>-0.2193</td>
</tr>
</tbody>
</table>

#### Inequality

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_e$</td>
<td>0.3972</td>
<td>0.3928</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$G_i$</td>
<td>0.3463</td>
<td>0.3614</td>
<td>$G_i/G_e$</td>
<td>0.8713</td>
</tr>
<tr>
<td>$G_c$</td>
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<td>0.2570</td>
<td>$G_c/G_e$</td>
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</tr>
<tr>
<td>$\sigma_{G_e}/\sigma_Y$</td>
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<td>1.8522</td>
<td>$\rho(Y,G_e)$</td>
<td>-0.5795</td>
</tr>
<tr>
<td>$\sigma_{G_i}/\sigma_Y$</td>
<td>0.9941</td>
<td>1.7488</td>
<td>$\rho(Y,G_i)$</td>
<td>-0.3241</td>
</tr>
<tr>
<td>$\sigma_{G_e}/\sigma_Y$</td>
<td>1.1233</td>
<td>1.2794</td>
<td>$\rho(Y,G_c)$</td>
<td>-0.2188</td>
</tr>
</tbody>
</table>

**Note:** $G_e$, $G_i$, and $G_c$ stand for the Gini coefficients for earnings, income, and consumption, respectively. The frequency of business cycle statistics is quarterly. Inequality statistics are annual, which are obtained by time-aggregating the simulation data from the quarterly model.
Table 6: The Role of Aggregate Shocks

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Both</th>
<th>z-Shock</th>
<th>s-Shock</th>
<th>Data</th>
<th>Both</th>
<th>z-Shock</th>
<th>s-Shock</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_Y )</td>
<td>1.6268</td>
<td>1.6408</td>
<td>1.5958</td>
<td>0.4736</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_C/\sigma_Y )</td>
<td>0.5022</td>
<td>0.5209</td>
<td>0.5254</td>
<td>0.4636</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_I/\sigma_Y )</td>
<td>2.7919</td>
<td>2.4129</td>
<td>2.3983</td>
<td>2.5757</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_H/\sigma_Y )</td>
<td>0.9510</td>
<td>0.8728</td>
<td>0.8841</td>
<td>0.4130</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_{Y/H}/\sigma_Y )</td>
<td>0.5816</td>
<td>0.4189</td>
<td>0.1226</td>
<td>1.4051</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho(Y, C) )</td>
<td>0.9242</td>
<td>0.9880</td>
<td>0.9886</td>
<td>0.9851</td>
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<td></td>
</tr>
<tr>
<td>( \rho(Y, I) )</td>
<td>0.9681</td>
<td>0.9951</td>
<td>0.9953</td>
<td>0.9960</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho(Y, H) )</td>
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<td>0.9088</td>
<td>0.9991</td>
<td>-0.9733</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho(Y, Y/H) )</td>
<td>0.9681</td>
<td>0.9951</td>
<td>0.9953</td>
<td>0.9960</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho(H, Y/H) )</td>
<td>-0.2193</td>
<td>0.0859</td>
<td>0.9383</td>
<td>-0.9866</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_{G_e}/\sigma_Y )</td>
<td>0.5607</td>
<td>1.8522</td>
<td>0.6716</td>
<td>5.8461</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_{G_i}/\sigma_Y )</td>
<td>0.9941</td>
<td>1.7488</td>
<td>0.7576</td>
<td>5.3316</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_{G_c}/\sigma_Y )</td>
<td>1.1233</td>
<td>1.2794</td>
<td>0.3137</td>
<td>4.3453</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The benchmark model, which is called ‘both’ in the table, includes both TFP shocks and dispersion shocks. The z-shock model keeps TFP shocks but eliminates dispersion shocks, while the s-shock model holds dispersion shocks but removes TFP shocks. \( G_e, G_i, \) and \( G_c \) stand for the Gini coefficients for earnings, income, and consumption, respectively. The frequency of business cycle statistics is quarterly. Inequality statistics are annual, which are obtained by time-aggregating the simulation data from the quarterly model.
<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Unemployment</th>
<th>Full Employment</th>
<th>Data</th>
<th>Unemployment</th>
<th>Full Employment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_Y$</td>
<td>1.6268</td>
<td>1.6408</td>
<td>1.4824</td>
<td>$\rho(Y, C)$</td>
<td>0.9242</td>
<td>0.9880</td>
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<tr>
<td>$\sigma_C/\sigma_Y$</td>
<td>0.5022</td>
<td>0.5209</td>
<td>0.5105</td>
<td>$\rho(Y, I)$</td>
<td>0.9681</td>
<td>0.9951</td>
</tr>
<tr>
<td>$\sigma_I/\sigma_Y$</td>
<td>2.7919</td>
<td>2.4129</td>
<td>2.4453</td>
<td>$\rho(Y, H)$</td>
<td>0.8234</td>
<td>0.9088</td>
</tr>
<tr>
<td>$\sigma_H/\sigma_Y$</td>
<td>0.9510</td>
<td>0.8728</td>
<td>0.5257</td>
<td>$\rho(Y, Y/H)$</td>
<td>0.3730</td>
<td>0.4938</td>
</tr>
<tr>
<td>$\sigma_{Y/H}/\sigma_Y$</td>
<td>0.5816</td>
<td>0.4189</td>
<td>0.6778</td>
<td>$\rho(H, Y/H)$</td>
<td>-0.2193</td>
<td>0.0859</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Unemployment</th>
<th>Full Employment</th>
<th>Data</th>
<th>Unemployment</th>
<th>Full Employment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_e$</td>
<td>0.3972</td>
<td>0.3928</td>
<td>0.3920</td>
<td>$-\rho(Y, G_e)$</td>
<td>-0.5795</td>
<td>-0.1049</td>
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<tr>
<td>$G_i$</td>
<td>0.3463</td>
<td>0.3614</td>
<td>0.3624</td>
<td>$G_i/G_e$</td>
<td>0.8713</td>
<td>0.9201</td>
</tr>
<tr>
<td>$G_c$</td>
<td>0.2994</td>
<td>0.2570</td>
<td>0.2577</td>
<td>$G_c/G_e$</td>
<td>0.7545</td>
<td>0.6544</td>
</tr>
<tr>
<td>$\sigma_{G_e}/\sigma_Y$</td>
<td>0.5607</td>
<td>1.8522</td>
<td>2.0608</td>
<td>$\rho(Y, G_e)$</td>
<td>-0.5795</td>
<td>-0.1049</td>
</tr>
<tr>
<td>$\sigma_{G_i}/\sigma_Y$</td>
<td>0.9941</td>
<td>1.7488</td>
<td>1.8888</td>
<td>$\rho(Y, G_i)$</td>
<td>-0.3241</td>
<td>-0.1730</td>
</tr>
<tr>
<td>$\sigma_{G_c}/\sigma_Y$</td>
<td>1.1233</td>
<td>1.2794</td>
<td>1.4898</td>
<td>$\rho(Y, G_c)$</td>
<td>-0.2188</td>
<td>0.0261</td>
</tr>
</tbody>
</table>

**Note:** The unemployment model is the benchmark model. The full employment model is the model eliminating the time-varying threshold of idiosyncratic labor productivity for employment from the benchmark model. $G_e$, $G_i$, and $G_c$ stand for the Gini coefficients for earnings, income, and consumption, respectively. The frequency of business cycle statistics is quarterly. Inequality statistics are annual, which are obtained by time-aggregating the simulation data from the quarterly model.
### Table 8: The Role of the Elasticity of Labor Supply

<table>
<thead>
<tr>
<th></th>
<th>Business Cycles</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
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<td>Data Low Unit High</td>
</tr>
<tr>
<td></td>
<td>(0.5) (1.0) (1.5)</td>
<td>(0.5) (1.0) (1.5)</td>
</tr>
<tr>
<td>( \sigma_Y )</td>
<td>1.6268 1.4608 1.6408 1.7606</td>
<td>( \rho(Y,C) ) 0.9242 0.9668 0.9880 0.9935</td>
</tr>
<tr>
<td>( \sigma_{C/Y} )</td>
<td>0.5022 0.4257 0.5209 0.5850</td>
<td>( \rho(Y,I) ) 0.9681 0.9931 0.9951 0.9961</td>
</tr>
<tr>
<td>( \sigma_{I/Y} )</td>
<td>2.7919 2.7188 2.4129 2.2198</td>
<td>( \rho(Y,H) ) 0.8234 0.8999 0.9088 0.9369</td>
</tr>
<tr>
<td>( \sigma_{H/Y} )</td>
<td>0.9510 0.9598 0.8728 0.8564</td>
<td>( \rho(Y,Y/H) ) 0.3730 0.3096 0.4938 0.5510</td>
</tr>
<tr>
<td>( \sigma_{Y/H} )</td>
<td>0.5816 0.4402 0.4189 0.3587</td>
<td>( \rho(H,Y/H) ) -0.2193 -0.1361 0.0859 0.2245</td>
</tr>
</tbody>
</table>

- **Note:** The model with unit elasticity is the benchmark model. \( G_e, G_i, \) and \( G_c \) stand for the Gini coefficients for earnings, income, and consumption, respectively. The frequency of business cycle statistics is quarterly. Inequality statistics are annual, which are obtained by time-aggregating the simulation data from the quarterly model.
<table>
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<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>0.90</th>
<th>0.99</th>
<th>1.00</th>
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<tr>
<td>$G_e$</td>
<td>0.4108</td>
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<td>0.3643</td>
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<td>0.2974</td>
<td>0.2961</td>
<td>0.2955</td>
<td>0.2954</td>
</tr>
<tr>
<td>$G_i$</td>
<td>0.3788</td>
<td>0.3716</td>
<td>0.3458</td>
<td>0.3274</td>
<td>0.2908</td>
<td>0.2746</td>
<td>0.2714</td>
<td>0.2704</td>
<td>0.2699</td>
<td>0.2699</td>
</tr>
<tr>
<td>$G_c$</td>
<td>0.2660</td>
<td>0.2572</td>
<td>0.2235</td>
<td>0.1998</td>
<td>0.1528</td>
<td>0.1364</td>
<td>0.1351</td>
<td>0.1349</td>
<td>0.1348</td>
<td>0.1348</td>
</tr>
<tr>
<td>$G_i/G_e$</td>
<td>0.9210</td>
<td>0.9161</td>
<td>0.9050</td>
<td>0.8986</td>
<td>0.9033</td>
<td>0.9106</td>
<td>0.9126</td>
<td>0.9132</td>
<td>0.9135</td>
<td>0.9136</td>
</tr>
<tr>
<td>$G_c/G_e$</td>
<td>0.6475</td>
<td>0.6340</td>
<td>0.5849</td>
<td>0.5483</td>
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<td>0.4541</td>
<td>0.4555</td>
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</tbody>
</table>

**Note:** $G_e$, $G_i$, and $G_c$ stand for the Gini coefficients for earnings, income, and consumption, respectively.
Figure 1: Gini Coefficient

(a) The Level of Gini Coefficient

(b) The Gini Coefficient of Income and Consumption Relative to Earnings

Source: Heathcote et al. (2010)
Note: The data are annual. The statistics are calculated over the common sample period from 1980 to 2004. The CPS is used for earnings and income, and the CEX is used for consumption. Consumption includes both nondurables and services.
Figure 2: Cyclical Pattern of Gini Coefficient

Source: Heathcote et al. (2010) and NIPA Tables

Note: The data are annual. The statistics are calculated over the common sample period from 1980 to 2004. The CPS is used for earnings and income, and the CEX is used for consumption. Consumption includes both nondurables and services. The logarithms of output and the Gini coefficients are detrended by the HP-filter with the smoothing parameter of 100.
Figure 3: Inequality and the Wealth Effect on Labor Supply

(a) The Level of Gini Coefficient

(b) The Gini Coefficient of Income and Consumption Relative to Earnings
Figure 4: Total Hours and Labor Market Responses to Aggregate Shocks

(a) Total Hours with TFP Shocks
(b) Labor Market with TFP Shocks
(c) Total Hours with Dispersion Shocks
(d) Labor Market with Dispersion Shocks

Note: In all the panels, the blue dashed curves describe steady states, while the red solid curves are the responses to TFP or dispersion shocks.
Figure 5: Labor Market Responses to the Combinations of Aggregate Shocks

Note: In all the panels, the blue dashed curves describe steady states, while the red solid curves are the responses to TFP or dispersion shocks.
Appendices

A The Proof of Propositions and Remarks

A.1 Proposition 1

To simplify notations, let \( y \equiv \log(x) \). Then, the idiosyncratic process becomes

\[
y' = \rho_x \left( \frac{s'}{s} \right) y + \frac{1}{2} \left( \frac{\rho_x s' s - (s')^2}{1 - \rho_x^2} \right) + s' \epsilon_x', \quad \epsilon_x' \sim N(0, 1).
\]

Suppose that current \( y \) is distributed as follows:

\[
y \sim N \left( -\frac{1}{2} \frac{s^2}{1 - \rho_x^2}, \frac{s^2}{1 - \rho_x^2} \right).
\]

Then, we need to show that

\[
y' \sim N \left( -\frac{1}{2} \frac{(s')^2}{1 - \rho_x^2}, \frac{(s')^2}{1 - \rho_x^2} \right).
\]

The mean of the distribution of \( y' \) is

\[
E(y') = \rho_x \left( \frac{s'}{s} \right) E(y) + \frac{1}{2} \left( \frac{\rho_x s' s - (s')^2}{1 - \rho_x^2} \right)
= \rho_x \left( \frac{s'}{s} \right) \left[ -\frac{1}{2} \frac{s^2}{1 - \rho_x^2} \right] + \frac{1}{2} \left( \frac{\rho_x s' s - (s')^2}{1 - \rho_x^2} \right)
= -\frac{1}{2} \frac{\rho_x s' s}{1 - \rho_x^2} + \frac{1}{2} \left( \frac{\rho_x s' s - (s')^2}{1 - \rho_x^2} \right)
= -\frac{1}{2} \frac{(s')^2}{1 - \rho_x^2}.
\]

In addition, the variance of the distribution of \( y' \) is

\[
Var(y') = \rho_x^2 \left( \frac{s'}{s} \right)^2 Var(y) + (s')^2
= \rho_x^2 \left( \frac{s'}{s} \right)^2 \left[ \frac{(s)^2}{1 - \rho_x^2} \right] + (s')^2
= \frac{\rho_x^2 (s')^2}{1 - \rho_x^2} + (s')^2.
\]
\[
\begin{align*}
\text{Since } x = \exp(y), \text{ the mean and variance of idiosyncratic labor productivity are} \\
E(x) &= \exp \left( E(y) + \frac{1}{2} \text{Var}(y) \right) = 1, \\
\text{Var}(x) &= E(x) \left[ \exp \left( \text{Var}(y) \right) - 1 \right] = \exp \left( \frac{s^2}{1 - \rho_x^2} \right) - 1.
\end{align*}
\]

Therefore, the proposition is proved. \(\blacksquare\)

### A.2 Proposition 2

Given the optimal decision of hours worked by households, the effective labor supply is

\[
L^S(w; s, z) = \int_{\hat{x}(w)}^{\infty} x h(x; w) f(x; s) dx
\]

\[
= \int_{\hat{x}(w)}^{\hat{x}(w)} x \left( \frac{wx}{\psi} \right)^{\frac{1}{\gamma}} f(x; s) dx + \int_{\hat{x}(w)}^{\infty} x h f(x) dx
\]

\[
= \left( \frac{w}{\psi} \right)^{\frac{1}{\gamma}} \int_{\hat{x}(w)}^{\hat{x}(w)} x^{1 + \frac{1}{\gamma}} f(x; s) dx + \bar{h} \int_{\hat{x}(w)}^{\infty} x f(x; s) dx
\]

where \(f(x; s)\) is the probability density function of idiosyncratic labor productivity. Now, let

\[
A(w; s, z) \equiv \int_{\hat{x}(s, z)}^{\hat{x}(w)} x^{1 + \frac{1}{\gamma}} f(x; s) dx,
\]

\[
B(w; s) \equiv \int_{\hat{x}(w)}^{\infty} x f(x; s) dx.
\]

Then, we have the following aggregate labor supply schedule:

\[
L^S(w; s, z) = \left( \frac{w}{\psi} \right)^{\frac{1}{\gamma}} A(w; s, z) + \bar{h} B(w; s).
\]

Therefore, the proposition is proved. \(\blacksquare\)
A.3 Remark 1

In order to show that the aggregate labor supply schedule is upward sloping in the market real wage $w$, it is sufficient to show that

$$
\left(\frac{w}{\psi}\right)^{\frac{1}{\gamma}} \frac{\partial A(w; s, z)}{\partial w} + \bar{h} \frac{\partial B(w; s)}{\partial w} = 0,
$$

because

$$
\frac{1}{\gamma} \frac{w}{\psi} \left(\frac{w}{\psi}\right)^{\frac{1}{\gamma}-1} A(w; s, z) > 0.
$$

Given the definitions of $A(w; s, z)$ and $B(w; s)$ in Proposition 2,

$$
\frac{\partial A(w; s, z)}{\partial w} = \hat{x}^{1+\frac{1}{\gamma}} f(\hat{x}; s) \frac{d\hat{x}(w)}{dw},
$$

and

$$
\frac{\partial B(w; s)}{\partial w} = -\hat{x} f(\hat{x}; s) \frac{d\hat{x}(w)}{dw}.
$$

Then, we have

$$
\left(\frac{w}{\psi}\right)^{\frac{1}{\gamma}} \frac{\partial A(w; s, z)}{\partial w} + \bar{h} \frac{\partial B(w; s)}{\partial w} = \left[\left(\frac{w}{\psi}\right)^{\frac{1}{\gamma}} - \bar{h}\right] \hat{x} f(\hat{x}; s) \frac{d\hat{x}(w)}{dw}.
$$

By the definition of $\hat{x} = (\psi/w)\bar{h}^\gamma$, the square bracket in the equation above is zero. Therefore, the aggregate labor supply schedule is upward sloping.

A.4 Remark 2

Given the distribution of idiosyncratic labor productivity, an increase in TFP should reduce the threshold of labor productivity for employment $\bar{x}(s, z)$ so that the unemployment rate declines. In other words,

$$
\frac{\partial \bar{x}(s, z)}{\partial z} < 0.
$$
The definition of $A(w; s, z)$ in Proposition 2 implies

$$\frac{\partial A(w; s, z)}{\partial z} = -\bar{x}^{1+\frac{1}{\gamma}} f(\bar{x}; s) \frac{\partial \bar{x}(s, z)}{\partial z} > 0.$$  

Since only $A(w; s, z)$ in the labor supply depend on TFP and $(w/\psi)^{1/\gamma}$ is positive, the aggregate labor supply rises whenever TFP increases.

\section*{B Detailed Computational Strategy}

\subsection*{B.1 Steady State of Economy}

The algorithm to find the steady state of the model economy is based on Ríos-Rull (1999). In the steady state, aggregate states and household distribution are time-invariant. Overall procedure is as follows. Firstly, set TFP at its steady state $\bar{z}$. In the model, $\bar{z} = 1$. Fix the steady state real interest rate at its target $\bar{r}$. The firm’s optimality conditions imply that $\bar{r}$ determines the steady state real wage as

$$\bar{w} = (1 - \alpha)\bar{z} \left( \frac{\alpha \bar{z}}{\bar{r} + \delta} \right)^{\frac{\bar{s}}{1 - \alpha}}.$$  

Next, choose all the parameter values except the average standard deviation of shocks to idiosyncratic labor productivity $\bar{s}$ and the subjective discount factor $\beta$. Set grid points for household asset holdings and idiosyncratic labor productivity, and set the transition probability matrix for the idiosyncratic process as in step (1). Now, given the steady state values for $(\bar{z}, \bar{r}, \bar{w})$, search $(\bar{s}, \beta)$ which match the target Gini coefficient of earnings and clear the capital market, respectively, by repeating step (1) to (3) below. Note that updating $\bar{s}$ is involved in step (1) because it affects the grid points of idiosyncratic labor productivity, while updating $\beta$ does not affect the grids.

\textbf{(1) Setting Grids} \hspace{1cm} N_a \text{ and } N_x \text{ grid points are set for household asset holdings } a \text{ and } x. \text{ I used } N_a = 1,000 \text{ and } N_x = 16. \text{ The grid for asset holdings covers the range of } [0.0001, 100]. \text{ It is unequally spaced. The grid is fine around the lower limit and is getting coarser as the points approach to the upper limit. Idiosyncratic labor productivity } x \text{ is discretized initially to } (N_x - 1) \text{ grid points with the associated transition probability}
matrix by Rouwenhorst’s (1995) method. Then the grid is extended to $N_x$ points and the transition probability matrix is adjusted accordingly.

One more grid point for idiosyncratic labor productivity is added for the steady state of economy to be consistent with the dynamic economy. In the dynamic economy, unemployment is time-varying and has the perfect negative correlation with TFP. TFP and unemployment are realized at one of three values. To generate the time-varying unemployment, there is a critical productivity, say $x_j$, among the initially selected $(N_x - 1)$ values for productivity \{${{x_1, x_2, ..., x_{N_x - 1}}}$\}. Regardless of TFP, all the households with productivity less than $x_j$ are unemployed, while all with productivity higher than $x_j$ are employed. The employment status of households with productivity $x_j$ depends on the realization of TFP. With high TFP, all of them are employed, but with low TFP, all are unemployed. Importantly, with medium TFP, half of the households with productivity $x_j$ are randomly employed and the rest of them are unemployed. In the steady state of economy, only medium state for TFP is realized, and the households with productivity $x_j$ has two types so that the critical productivity $x_j$ needs two points: one for the employed and the other for the unemployed.

(2) Modified Policy Function Iteration with $k$ Steps Given $\bar{s}$ and $\beta$, the steady state version of household problem is solved to find the associated policy functions as follows.

a. Guess initial value function $V_0(a, x)$. For example, $V_0(a, x) = 0$.

b. Solve the following household problem to have the value function $V_{0,1}(a, x)$ and the policy function $c(a, x)$, $h(a, x)$, and $a'(a, x)$.

$$V_{0,1}(a, x) = \max_{\{c, h, a'\}} \left\{ \log \left( c - \psi \frac{h^{1+\gamma}}{1+\gamma} \right) + \beta \sum_{x'} \pi_x(x'|x)V_{0,0}(a', x') \right\},$$

where, $\pi_x(x'|x)$ is the transition probability from $x$ to $x'$. Whenever $a'$ is out of grid points, $V_0(a', x')$ is evaluated with the linear interpolation.

c. Repeat the following from $j = 1$ to $j = (k - 1)$ to have the value function $V_{0,k}$.

$$V_{0,j+1}(a, x) = \log \left( c(a, x) - \psi \frac{h(a, x)^{1+\gamma}}{1+\gamma} \right) + \beta \sum_{x'} \pi_x(x'|x)V_{0,j}(a'(a, x), x')$$

d. Set $V_1(a, x) = V_{0,k}(a, x)$. 44
e. If $V_0(a, x)$ and $V_1(a, x)$ are close enough, then go to step (3). Otherwise, update the value function as $V_0(a, x) = V_1(a, x)$ and go to step (2)b.

(3) **Limit Distribution** Given the policy function $a'(a, x)$ from step (2), the limit distribution of households is obtained as follows.

a. Initialize the household distribution $\mu_0(a, x)$ over the rectangular grid which is set at step (1).

b. Use the policy function $a'(a, x)$ to have $\mu_1(a, x)$ as follows: If $a_i$ and $a_{i+1}$ are two sequential grid points for asset holdings and $a_i \leq a'(a, x) < a_{i+1}$, then, for all $j \in \{1, 2, ..., N_x\}$,

$$
\mu_1(a_i, x_j) = \mu_0(a, x) \pi_x(x_j|x) \omega,
$$

$$
\mu_1(a_{i+1}, x_j) = \mu_0(a, x) \pi_x(x_j|x)(1 - \omega),
$$

where $\omega = (a_{i+1} - a'(a, x))/(a_{i+1} - a_i)$.

c. If $\mu_0(a, x)$ and $\mu_1(a, x)$ are close enough, then let the limit distribution be $\bar{\mu}(a, x) = \mu_1(a, x)$ and go to step (4). Otherwise, go to step (3)b with $\mu_0(a, x) = \mu_1(a, x)$.

(4) **Updating $\beta$** With the policy function $h(a, x)$ from step (2) and the limit distribution $\bar{\mu}(a, x)$ from step (3), the implied real interest rate is calculated as follows:

$$
r(\beta; \bar{s}) = \alpha \bar{z} \left( \frac{K(\beta; \bar{s})}{L(\beta; \bar{s})} \right)^{1-\alpha} - \delta,
$$

where

$$
K(\beta; \bar{s}) = \sum_a \sum_x a \bar{\mu}(a, x), \quad \text{and} \quad L(\beta; \bar{s}) = \sum_a \sum_x x h(a, x) \bar{\mu}(a, x).
$$

If $r(\beta; \bar{s})$ is close enough to $\bar{r}$, then the limit distribution is regarded as the steady state distribution given $\bar{s}$, i.e., $\mu^*(a, x) = \bar{\mu}(a, x)$, and go to step (5). Otherwise, go to step (2) with a new $\beta$. Because $r(\beta; \bar{s})$ is monotonically decreasing in $\beta$, a root finding method such as bisection search can be used to find $\beta$ satisfying $r(\beta; \bar{s}) = \bar{r}$.

(5) **Updating $\bar{s}$** Given the real wage $\bar{w}$, policy function $h(a, x)$, and steady state distribution $\mu^*(a, x)$, calculate the Gini coefficient for earnings. If it is close enough to the target
Gini coefficient, then the solution for the steady state of economy and the parameter values for $\bar{s}$ and $\beta$ are found. Otherwise, go to step (1) with a new $\bar{s}$.

### B.2 Dynamic Economy

The algorithm for the dynamic economy extends the method proposed by Krusell and Smith (1998). As in the steady state of economy, the household state is described by household asset holdings and idiosyncratic labor productivity. In addition to the individual states, the current aggregate state of economy is described by three aggregate variables: the distribution of households, TFP, and the dispersion of labor productivity. Among the aggregate state variables, the distribution of households $\mu$ is the infinite dimensional element, which makes it impossible to solve the model. For the feasibility of the numerical analysis, the distribution must be replaced with a finite dimensional element. As Krusell and Smith (1998), it is assumed that the agents in the model have limited information on the household distribution. For the distribution of asset holdings or wealth, the agents are assumed to be informed of only its first moment or the average of asset holdings. Then, the productivity distribution becomes redundant information because the dispersion state, which is already included as one of the state variables, reveals the marginal distribution of idiosyncratic labor productivity. Therefore, the household distribution is approximated by the average of asset holdings or aggregate capital $K$. Accordingly the transition operator $\Gamma(\Omega, s')$ of household distribution is replaced with a log-linear forecasting rule for the future aggregate capital:

$$\log(K') = \beta_0 + \beta_1 \log(K) + \beta_2 \log(z) + \beta_3 \log(s).$$

The procedure to find the dynamic equilibrium is as follows.

(1) Setting Grids  

The grids for the five state variables and the transition probability matrices for three exogenous processes are chosen. The grid for asset holdings are set over the range of $[0.0001, 100]$ with $N_a$ points. As in the steady state computation, the grid is unequally spaced with more grid points for lower asset holdings. $N_k$ grid points for aggregate capital are equally spaced over $[0.7K^*, 1.3K^*]$ where $K^*$ is the steady state aggregate capital. I use $N_a = 75$ and $N_k = 7$.

TFP and dispersion processes are discretized to $N_z$ and $N_s$ grid points, respectively. Their grids and transition probability matrices are chosen using the method by Rouwenhorst (1995). I assume $N_z = 3$ and $N_s = 3$. The idiosyncratic labor productivity process is
set by the same way as in the steady state computation with $N_x = 16$. In the dynamic economy, however, the idiosyncratic process is involved in the dispersion state, and so is its discretization. As explained below, the transition probability matrix for idiosyncratic process is invariant to the dispersion state. Only grids are need to be set for each dispersion state. Therefore, the transition probability matrix for the idiosyncratic process is unique, but $N_s$ sets of grid are chosen for the idiosyncratic labor productivity.

More specifically, it is convenient to separate the idiosyncratic labor productivity process (3) into two steps as follows.

\begin{align*}
\text{(step 1: adjusting distribution)} & \quad \log(\tilde{x}) = \left( \frac{s'}{s} \right) \left( \log(x) + \frac{1}{2} \frac{s^2}{1 - \rho_x^2} \right) - \frac{1}{2} \frac{(s')^2}{1 - \rho_x^2} \\
\text{(step 2: shuffling productivity)} & \quad \log(x') = \rho_x \log(\tilde{x}) - \frac{1}{2} \frac{(s')^2}{1 + \rho_x} + s' \epsilon'_x, \quad \epsilon'_x \sim N(0, 1)
\end{align*}

Once the future dispersion state $s'$ realized, firstly, idiosyncratic labor productivity is adjusted to $\tilde{x}$ from $x$ so that the distribution of $\tilde{x}$ can accommodate the new limit distribution of $x'$ while keeping the order of household productivity. In the next step, the idiosyncratic shocks shuffle labor productivity across households.

Between the two steps, only the second step matters for the discretization. Under Rouwenhorst’s (1995) method, only the persistence parameter determines the transition probability matrix. Since the persistence is not affected by the dispersion state in the second step, the transition probability matrix is unique given the persistence. On the other hand, the grids of idiosyncratic labor productivity depend on the dispersion state. This implies that, under the discretization, the probability mass function is unique over the indices of the grid for labor productivity regardless of the dispersion state, although the indices refer to different productivity depending on the dispersion state.

\section*{(2) Factor Prices}

For each aggregate state $\hat{\Omega} \equiv \{K, s, z\}$, the real wage $w$ and the real interest rate $r$ are found. Because idiosyncratic labor productivity process is discretized, the aggregate labor supply schedule in Proposition 2 is modified accordingly by replacing the probability density with probability mass function as follows:

\[
L^S(w; s, z) = \left( \frac{w}{\psi} \right)^\frac{1}{q} \tilde{A}(w; s, z) + \tilde{h}\tilde{B}(w; s),
\]
where
\[ \hat{A}(w; s, z) = \sum_{x(s) \leq x_i(s) \leq x(w)} x(s)^{1+\frac{1}{\gamma}} m_i, \quad \text{and} \quad \hat{B}(w; s) = \sum_{x_i(s) > x(w)} x_i(s)m_i, \]
and \( m_i \) is the probability mass for the productivity index \( i \in \{1, 2, ..., N_x\} \). As mentioned above, due to the characteristic of Rouwenhorst’s (1995) method, the probability mass over the productivity index is unique regardless of the dispersion state but actual household productivity for each index varies with the dispersion state.

The real wage function \( w(\hat{\Omega}) \) is obtained by a root finding technique, for example bisection search, which clears the labor market given the aggregate labor demand schedule (7) and the aggregate labor supply schedule (11). Then, the real interest rate is determined as follows:
\[ r(\hat{\Omega}) = \alpha z \left( \frac{(1 - \alpha)z}{w(\hat{\Omega})} \right)^{\frac{1-\alpha}{\alpha}} - \delta. \]

(3) Modified Policy Function Iteration with k Steps

Given the factor prices \( w(\hat{\Omega}) \) and \( r(\hat{\Omega}) \) from step (2), and the forecasting rule (10) with the coefficients \( B^0 = \{\beta_0^0, \beta_1^0, \beta_2^0, \beta_3^0\} \), the household problem (1) is solved to have the associated policy functions. This step is analogous to step (2) in the computation for the steady state.

(4) Simulation

Given the policy functions from step (3), the model economy is simulated to have sufficiently long time-series for the aggregate capital. For quantitative results, the simulation data of 100,000 households for 3,500 periods are generated. The ordinary least square (OLS) estimation with the last 3,000 data are used to update the set of coefficients in the forecasting rule (10) to \( B^1 = \{\beta_0^1, \beta_1^1, \beta_2^1, \beta_3^1\} \).

In every period, the market clearing factor prices are calculated. This is necessary because the aggregate capital is continuously varying over time in the simulation so that it is out of grid points over which factor price functions are obtained. Therefore, we cannot use the factor prices in step (2). Finding the market clearing prices are not computationally costly because it is not involved in solving a optimization problem. This is not the case if the preferences imply non-zero wealth effects on labor supply. (to compare, see Khan and Thomas (2003, 2008), and Takahashi (2012) among others).

If \( B^0 \) and \( B^1 \) are close enough, go to step (5). Otherwise, go to step (3) with the updated
set of coefficients $B^0 = B^1$.

(6) Accuracy of Forecasting Rule The accuracy of the forecasting rules are checked by $R^2$. If it is sufficiently high, then the dynamic equilibrium is found. Otherwise, go to step (3) with a different specification for forecasting rule, for example by adding other variables such as the second moments to the regressors.

It turns out that the forecasting rule (10) is quite accurate. For the benchmark model, the converged forecasting rule is

$$\log(K') = 0.0697 + 0.9801 \log(K) + 0.0963 \log(z) + 0.0090 \log(s),$$

with $R^2 = 0.999967$ and the standard error of $1.3185e-4$. 
